Heat, Air and Moisture

in

Building Envelopes

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Preface

Building spaces are separated from each other and from the outdoor climate by partitions: facades, roofs, floors, inside walls. These structures are subjected to a strongly varying climate: sunshine, rain, wind and air temperatures. The performance requirements put on these structures depend on the requirements for comfort in the rooms of a building, the energy needed to get the desired indoor climate, the air quality and air humidity of this climate, the durability, maintenance, use of materials and the recyclebility of these structures.

In the past the design of these structures was lead by experience. By the rigid requirements on performance and the enormous increase of new building techniques, new materials and new building shapes this is often not applicable any more. The result is building damage, a bad indoor climate and an unnecessary large energy use. Therefore the knowledge of heat and moisture transport through building structures and joints is increasingly more important for building design. A clear illustration of this fact is the Dutch building code (bouwbesluit) with its abundant regulations related to building physics. The knowledge, insight and prediction models of building physics are indispensable for the realisation of high quality buildings that satisfy the required performances.
0. Introduction

The view that heat is a form of kinetic energy is only just after the experiments of Rumford, Davy and Joule (1818 - 1889) recognised as the only right one. In 1840 Joule started with the experimental determination of the quantity of mechanical energy (force x path) that is equivalent with the quantity of heat (1 calorie: heat needed to heat 1 gram of water from 14½ to 15½ °C). In the SI-system the same unity is used for both types of energy, the joule, (1 calorie = 4.184 joule). For energy just as for mass and momentum a conservation law is valid.

The conservation of energy is called the first law of thermodynamics:

\[ \text{The amount of thermal and mechanical energy that enters a control volume plus the amount of thermal energy that is generated within the volume is equal to the amount of thermal and mechanical energy that leaves the volume plus the increase of energy stored in the control volume.} \]

In building physics the stored thermal energy usually equals the so-called internal energy that consists of the kinetic energy of the molecules. For gases also the mechanical energy resulting from a volume change (thermal expansion) at constant pressure (volume-work) and condensation of water vapour (latent heat) has to be considered.

The property that equals the sum of the internal energy, the latent heat and the volume-work is called the enthalpy. So the total supply of enthalpy equals the increase of enthalpy of a system.

Other forms of mechanical energy are in building physics only important as far as they are dissipated (converted to heat).

We shall refer to the first law as the heat balance of a control volume.

Three mechanisms to transport heat will be treated:

- conduction
- convection
- radiation

Conduction is the energy transfer by molecules that from a macroscopic point of view don’t move compared to each other. By this energy transfer free electrons play an important part. This explains that most good electrical conductors are also good thermal conductors. As atomic nuclei also contribute significantly to energy transfer there are in solids no thermal insulators analogous to electrical insulators.
Energy transfer by convection is caused by the uptake of energy by the fluid at a certain place and delivery at a different place. If the flow is turbulent the convective heat transfer is the dominating heat transfer mechanism. In case of a laminar flow the heat transfer perpendicular to the flow direction is mainly by conduction.

Energy transfer by radiation is caused by emission and absorption of electromagnetic waves by materials. At emission heat is converted into electromagnetic energy and at absorption the reverse happens. The waves are propagated in vacuum without absorption and at the distances that are relevant in the built environment absorption by air can be neglected. In the temperature range that is relevant here i.e. between -20°C and +40°C, the radiation emitted by material surfaces is called thermal or heat radiation.

In building physics mass transfer is above all air and moisture flow. Airflows are caused by temperature differences (buoyancy), wind and appliances (ventilators).

Moisture appears in porous building materials as water, vapour and in a more or less bounded form. The absorption of moisture from the environment in the pores at a low relative humidity is caused by surface condensation and at a higher relative humidity by capillary condensation at the curved water meniscus in the pores.

The main mechanisms for moisture transfer are:

- flow by pressure differences
- advection
- diffusion

Pressure differences can originate from wind pressures, gravity (water pressure) and capillary suction.

The first description of the transfer of water by capillary suction and gravitation in a porous medium can be found in the publication of Darcy (1856): "Les fontaines publiques de la ville de Dijon".

Advection is the dragging along of water vapour and water droplets by another flowing medium; in building physics this medium is usually air. It is also called convective transfer.

In a porous material vapour is mainly transferred by diffusion. This diffusion is caused by movements of molecules in the air (molecular diffusion). Diffusion attempts to level out concentration differences (differences in vapour pressure).

The problem of moisture transfer in porous materials is more or less similar to the problem of heat transfer. Both are diffusion processes. This similarity will be exploited as much as possible.
In most problems heat transfer by radiation, convection and conduction occur jointly, e.g. at the surface of a material. The same happens in the pores of a material. For porous materials the effect of radiation, convection and conduction is accounted for in the (equivalent) thermal conductivity.

In addition to the combination of the three modes of heat transfer there is in building structures also a coupling of heat, air and moisture transfer: air flows cause heat and moisture transfer, temperature differences cause differences in relative humidity’s that in turn cause moisture transfer, evaporation and condensation depend on temperature and influence the heat balance (latent heat).

Fortunately for building physics many simplifications are permitted because the processes happen at atmospheric pressure, at a relatively small temperature range and low flow velocities.
1. The equations for heat and mass transfer

1.1. The heat and mass balance

1.1.1. The balance of a control volume

The heat balance reads:

For a control volume and a definite interval of time the sum of the heat that enters the volume and the heat dissipated in the volume equals the sum of the quantity leaving the volume and the energy stored in the volume.

In the form of an equation:

\[ \Phi_{\text{in}}(t) + \Phi_p(t) = \Phi_{\text{out}}(t) + \frac{d}{dt} Q_{\text{acc}} \]  

(1.1)

where:

- \( \Phi_{\text{in}}(t) \) = the heat flow rate to the volume as a function of time [W]
- \( \Phi_{\text{out}}(t) \) = the heat flow rate from the volume as a function of time [W]
- \( \Phi_p(t) \) = the dissipated quantity of heat per unit of time [W]
- \( Q_{\text{acc}} \) = the stored quantity of heat [J]

heat loss calculation

In a heat loss calculation (the calculation of the maximum power needed for a heating plant) \( \Phi_{\text{out}} - \Phi_{\text{in}} \) is the sum of ventilation and conduction losses and \( \Phi_p \) is the heating power delivered by the plant. The storage term is not taken into account explicitly but implicitly by using an outdoor temperature that is higher than the minimum occurring temperature (the mean temperature over a longer time) and additionally by adding an extra heating power for a short heating up time.

heating demand calculation

The (manual) calculation of the heating demand is not about the heat flow rate [W] but the quantity of heat [J]. The heat balance in this case is:

\[ Q_{\text{in}} + Q_p = Q_{\text{out}} \]  

(1.2)

Where \( Q_{\text{out}} - Q_{\text{in}} \) is the sum of the ventilation and conduction losses [J] and \( Q_p \) the sum of the internal heat sources, the absorbed solar radiation and the heat (auxiliary heating) delivered by the heating plant.

The storage of heat is not present any more in eq.1.2 but has certainly influenced the difference \( Q_{\text{out}} - Q_{\text{in}} \); storage decreases the number of hours the temperature exceeds the desired value by solar radiation.
When the temperature rises to uncomfortable high values by solar gains there is an excess of heat that cannot be considered as a part of the heating demand. Inhabitants often will increase the heat loss then e.g. by opening a window. This excess of heat is subtracted from the solar gains resulting in a lower solar utilisation factor. Thus more heat storage increases the solar utilisation factor.

At night setback the temperature will decrease less by heat storage. So the average indoor temperature is higher and the result is more heat loss compared with a room with less storage. So the energy savings by night setback are smaller by energy storage.

The equation for the mass flows can be derived in the same way as eq.1.1:

\[ G_{in} + G_p = G_{out} + \frac{dm}{dt} \]  

(1.3)

where:  
- \( G_{in} \) = the mass flow rate to the volume [kg/s]  
- \( G_{out} \) = the mass flow rate from the volume [kg/s]  
- \( G_p \) = the mass source [kg/s]  
- \( m \) = the quantity of mass [kg]

An example of this is the moisture balance for water vapour in a room where \( G_{out} - G_{in} \) is the net flow rate of water vapour by ventilation, \( G_p \) the mass of the vapour that is released per unit of time in the room (e.g. by evaporation of water) and \( m \) the stored mass of moisture in the room (e.g. in the room air).

Note that what is denoted here as the heat and mass balance equations are equations that follow from the conservation of energy and of mass.

For a small interval of time \( \Delta t \) the heat balance is:

\[ \Phi_{in}(t)\Delta t + \Phi_p(t)\Delta t = \Phi_{out}(t)\Delta t + \Delta Q_{acc}(t) \]

with: \( \Delta Q_{acc} \) = the increase of the amount of stored heat

After dividing by \( \Delta t \) and by taken the limit for \( \Delta t \to 0 \) one gets the differential form (eq.1.1)

In order to get a balance for heat quantities the equation has to be integrated over a certain period (e.g. the heating season)

If the length of this period is \( t_2 - t_1 \), then:

\[ Q_{in} + Q_p = Q_{out} + (Q_{acc}(t_2) - Q_{acc}(t_1)) \]

When the period \( t_2 - t_1 \) is longer the quantities in the equation, with the exception of the storage term, will be larger: the heat storage in the building at \( t_1 \) is not necessarily different from the storage at \( t_2 \). So the storage term is relatively smaller.
1.1.2. Heat and mass conservation in a point

The heat flow rate has a direction. For example, heat can be carried along by airflow and has in that case the direction of that flow or heat is transferred from a high temperature to a lower one. Furthermore, the heat flow rate will depend on the size of the section through which the heat flows, i.e., a density of heat flow rate \( q \) \([W/m^2]\) can be defined as:

\[
q = \frac{\Delta \Phi}{\Delta A}
\]

where: \( \Delta A \) = area of infinitesimal section perpendicular to the direction of the heat flow rate \([m^3]\), 
\( \Delta \Phi \) = the heat flow rate \([W]\) through \( \Delta A \).

So just as the heat flow rate the density of heat flow rate has a direction; it is a vector. The vector is denoted by \( q = (q_x, q_y, q_z) \) where \( q_x, q_y \) and \( q_z \) are the co-ordinates in an orthogonal (Cartesian) axis system.

The magnitude of the stored heat in a material will depend on the volume of that material. So also here a density \( e \) \([J/m^3]\) can be introduced according to:

\[
e = \frac{\Delta Q}{\Delta V}
\]

with \( \Delta V \) = an infinitesimal volume \([m^3]\) and \( \Delta Q \) the stored heat in it.

In the same way a heat source density \( S_e \) \([W/m^3]\) can be defined:

\[
S_e = \frac{\Delta \Phi}{\Delta V}
\]

Conservation of energy (eq. 1.1) expressed in these quantities reads:

\[
\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} + S_e = \frac{\partial e}{\partial t}
\]

The notation \( \partial \) instead of \( d \) is used because the derivative is taken from one variable (e.g., only \( t \)) while the function (e.g., \( e \)) depends on four variables (\( x, y, z, t \)) (partial differentiation). The differential equation above applies for each point in space instead of for a control volume as eq. 1.1.

An alternative notation is:

\[
\text{div } q + S_e = \frac{\partial e}{\partial t} \quad \text{div } = \text{divergence}
\] (1.4)

Similarly, a density of mass flow rate \( \vec{g} \) \([kg/m^2s]\), mass density \( w \) \([kg/m^3]\) and source density \( S_m \) \([kg/m^3s]\) can be defined. Mass conservation (eq. 1.3) then reads:

\[
- \text{div } \vec{g} + S_m = \frac{\partial w}{\partial t}
\] (1.5)
The symbols ‘w’ and  are customary for moisture content (mass by volume) of a material and (vector) density of moisture flow rate respectively. Equation 1.5 applies of course also for the flow of air through a porous material. In that case we shall add the subscript ‘a’ (air).

The equations 1.4 and 1.5 cannot be solved yet, because apart from the lack of boundary and initial conditions there are also too many unknown quantities. These quantities have to be expressed in but two quantities as there are two equations. These quantities are called potentials or state variables. In general the r.h.s. of eq.1.4 and eq.1.5 are related to state variables with the so-called capacitive properties of materials using irreversible thermodynamics. For the l.h.s. phenomenological equations exist (often called ‘law’) which relate flows to changes of state variables per unit of distance in the direction of the flows (gradient). The proportionality factor in the equations is a material property and is called the transfer coefficient.

By the transfer of energy or mass there is no thermodynamic equilibrium. As the thermodynamic state changes slowly quasi-equilibrium can be assumed i.e. it is allowed to use for the storage term the thermodynamic relation valid for equilibrium.

The potential for the heat is the thermodynamic temperature [K], and for moisture in the form of vapour, the vapour pressure and for water in capillary materials it is the capillary suction. Other choices are also possible depending on the problem at hand and the conceivable simplifications. In the next sections this will be dealt with.

For an infinitesimal block (control volume) $\Delta V = \Delta x \Delta y \Delta z$ the terms of the heat balance are (with the centre of gravity in the point x, y, z):

$$
\Phi_{in} = \Delta y \Delta z q_x (x-\Delta x/2, y, z, t) + \Delta x \Delta z q_x (x, y-\Delta y/2, z, t) + \Delta x \Delta y q_y (x, y, z-\Delta z/2, t)
$$

$$
\Phi_{out} = \Delta y \Delta z q_x (x+\Delta x/2, y, z, t) + \Delta x \Delta z q_x (x, y+\Delta y/2, z, t) + \Delta x \Delta y q_y (x, y, z+\Delta z/2, t)
$$

$$
Q_{acc} = \Delta x \Delta y \Delta z \cdot e(x, y, z, t)
$$

After division of this heat balance by $\Delta x \Delta y \Delta z$ and using the definition for a partial derivative

$$
q_x (x+\Delta x/2, y, z, t) - q_x (x-\Delta x/2, y, z, t) = \frac{\partial q_x}{\partial x}
$$

and similarly $\partial q_y/\partial y$ and $\partial q_z/\partial z$, we get equation 1.4.
1.2. The heat conduction equation

1.2.1. Fourier’s law

In general the temperature in a material can be different every instant and on every place. Thus \( T = T(x,y,z,t) \) where \( x,y,z \) are the co-ordinates of the point considered in an orthogonal axis system and \( t \) the time.

In the material heat will flow from a higher to a lower temperature. According to Fourier the density of heat flow rate in an isotropic material (material properties are independent of the direction) is proportional to the temperature decline per unit of length:

\[
\begin{align*}
  q_x &= -\lambda \frac{\partial T}{\partial x}, \\
  q_y &= -\lambda \frac{\partial T}{\partial y}, \\
  q_z &= -\lambda \frac{\partial T}{\partial z}
\end{align*}
\]  \( (1.6) \)

with \( \lambda = \text{thermal conductivity} \ [\text{W/mK}] \)
\( T = \text{thermodynamic temperature} \ [\text{K}], \ \Theta = T-273.15 \ [\text{°C}] \)

The \(-\) sign is caused by the agreement that the heat flow rate is positive for a declining temperature. In a different notation:

\[ q = -\lambda \text{ grad } T \quad \text{grad = gradient} \]

The gradient of a function is a vector: \( \text{grad} f = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \)

In an anisotropic material the thermal conductivity can be different in different directions (e.g. wood). As a consequence the direction of the gradient vector \( (\partial T/\partial x, \partial T/\partial y, \partial T/\partial z) \) doesn’t coincide with the heat flow vector \( (\text{e.g. } \lambda_x \partial T/\partial x, \lambda_y \partial T/\partial y, \lambda_z \partial T/\partial z) \) any more.

The heat needed to increase the temperature of 1kg material 1°C is called the specific heat capacity \( c \) (if no phase change occurs). So for 1 m\(^3\) this heat is \( \rho c \) [J]. \( \rho c \) is called the volumetric heat capacity [J/m\(^3\)K].

So a change of the stored heat per unit of volume is proportional to the change in temperature:

\[ \Delta e = \rho c \Delta T \]

where: \( c = \text{specific heat capacity} \ [\text{J/kgK}] \)
\( \rho = \text{density} \ [\text{kg/m}^3] \)

The heat capacity depends on the circumstances during heating. If the pressure is constant the value of this property will differ from the value of a situation with a constant volume, as energy is needed for the volume change (volume-work). In building physics the pressure can be considered constant. So the specific heat capacity at a constant pressure \( (c_p) \) is used most often. Solids and fluids have about the same value regardless of the circumstances during heating so the subscript \( p \) is superfluous. Inserting this and Fourier’s law (eq.1.6) in the mass balance equation (eq.1.4) yields:
\[ \rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( \lambda \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \lambda \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( \lambda \frac{\partial T}{\partial z} \right) \]

or: \[ \rho c \frac{\partial T}{\partial t} = \text{div}\lambda \text{ grad } T \]

\[ \text{div} = \text{divergence} \] (1.7)

The divergence of a vector is: \[ \text{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x} f_x + \frac{\partial}{\partial y} f_y + \frac{\partial}{\partial z} f_z \]

In general the thermal conductivity depends on the temperature and the moisture content of a material (is a function of \(T(x,y,z,t)\) and \(w(x,y,z,t)\)). In practice a constant value is most of the time sufficient accurate.

Cylindrical bodies with rotation symmetry often have temperature gradients in radial (r) and axial directions (z) only. So they are 2-D. In cylindrical co-ordinates the heat conduction equation is:

\[ q_r = -\lambda \frac{\partial}{\partial r} T, \quad q_z = -\lambda \frac{\partial}{\partial z} T \]

\[ \rho c \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( \lambda \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial z} \left( \lambda \frac{\partial T}{\partial z} \right) \]

Suppose a problem has a characteristic length \(d\) and one is interested in solutions that have a characteristic time step \(\Delta t\) (e.g. one hour). With the dimensionless time \(\tau = t/\Delta t\) and the dimensionless spatial co-ordinates \(\chi = x/d\), etc the equation reads:

\[ \frac{\partial T}{\partial \tau} = \text{Fo} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \]

with: \[ \text{Fo} = a \Delta t / d^2 \] the Fourier number [-]
\[ a = \lambda / \rho c \] the thermal diffusivity \([m^2/s]\)

The Fourier number is a measure for the ratio heat conduction and change of heat storage. If the Fourier number is large the heat storage has hardly any influence.

<table>
<thead>
<tr>
<th>Material</th>
<th>(c) [J/kgK]</th>
<th>(\rho c) [J/m^3K]</th>
<th>(\lambda) [W/mK]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Metals</td>
<td>130 –180</td>
<td>&gt;300\times10^4</td>
<td>16-380</td>
</tr>
<tr>
<td>Stony materials</td>
<td>840</td>
<td>&gt;150\times10^4</td>
<td>0.15-3.5</td>
</tr>
<tr>
<td>Polymers</td>
<td>1470</td>
<td>&gt;250\times10^4</td>
<td>0.1-0.25</td>
</tr>
<tr>
<td>Wood, wood product</td>
<td>1880</td>
<td>&gt;70\times10^4</td>
<td>0.1-0.25</td>
</tr>
<tr>
<td>Mineral wool</td>
<td>840</td>
<td>&gt;1.5\times10^4</td>
<td>0.04</td>
</tr>
<tr>
<td>Water</td>
<td>4200</td>
<td>420\times10^4</td>
<td>0.6</td>
</tr>
<tr>
<td>Water vapour 0°C&lt;(\Theta)&lt;50°C</td>
<td>1860</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Air</td>
<td>1000</td>
<td>0.12\times10^4</td>
<td>0.025</td>
</tr>
</tbody>
</table>
1.2.2. The initial and boundary conditions

The initial condition is the temperature distribution at the beginning of the period for which a calculation is made. For a steady-state problem (no heat storage) the initial condition is superfluous: the solution has no time dependency, the problem has no ‘memory’. Very often the initial condition is not known and a reasonable estimate has to be made. Only after some time the solution will be reliable, e.g. at the start of a computer simulation of the annual heating demand of a building the temperatures in the walls are not known and have to be estimated. In general the calculation results of the first three days are not reliable. After three days the influence of the first day is negligible (the building has ‘forgotten’ the first day).

Building envelope parts and partitions consist of assemblies of different materials. For each material the heat conduction equation applies but until now the boundary conditions at the interface between these materials or between a material and the air is missing.

a. The boundary conditions at the interface between two materials

The density of heat flow rate perpendicular to the interface is continuous (has no jump). This condition follows from the heat balance of a thin layer parallel to the interface and consisting of both materials at the interface. If the z-axis is taken perpendicular to the interface, the boundary condition is:

$$ q_{z1} = q_{z2} \text{ of } -\lambda_1 \left( \frac{\partial T}{\partial z} \right)_{1} = -\lambda_2 \left( \frac{\partial T}{\partial z} \right)_{2} $$

(1.8)

where: $q_{z1} =$ the density of heat flow rate in material 1(⊥ interface)
$q_{z2} =$ idem in material 2.

The directions of the density of heat flow rate vector in material 1 and 2 don’t need to be the same because $q_x$ and $q_y$ don’t need to be equal.

Usually an ideal thermal contact is assumed between the two materials:

$$ T_1 = T_2 $$

(1.9)

where: $T_1$, $T_2$ = the temperatures at the interface surface in material 1 and 2 respectively

At very high heat flux densities this is not valid any more and this assumption can lead to considerable errors. An extra contact resistance has to be introduced.

b. Boundary conditions in the ground

For structures bordered by the ground, a layer of soil has to be included in the calculation. The thickness of this layer must be such that a constant temperature or an adiabatic (no density of heat flow rate) boundary condition can be assumed. For example the vertical boundary
conditions in the ground (see left picture of a foundation calculation detail) are adiabatic and the horizontal boundary condition (bottom) is:

\[ T_w = T_b \]  

(1.10)

with \( T_b \) = the temperature deep in the ground
(e.g. \( \Theta_b = 10^\circ C \), 3m below ground level).

c.  \textit{The boundary conditions at a surface to the air}

For a surface bordering the air the density of heat flow rate perpendicular to the surface is continuous as well. This density at the airside is:

\[ q_z = h_{cv} (T_a - T_s) + h_r (T_r - T_s) + \sum_j \alpha_j E_j \cdot \varepsilon \Delta E_{at} \]

where:
- \( h_{cv} \) = the surface coefficient of convective heat transfer [W/m²K]
- \( h_r \) = the surface coefficient of radiative heat transfer [W/m²K]
- \( T_a \) = the surface temperature
- \( T_s \) = the air temperature
- \( T_r \) = the 'mean' radiant temperature perceived by the surface
- \( \alpha_j \) = absorptivity of the surface for radiation of source j [-]
- \( E_j \) = the irradiance of radiant source j (sun, lighting) [W/m²]
- \( \varepsilon \) = emissivity of the surface [-]
- \( \Delta E_{at} = \sigma T_e^4 - E_{at} \) with \( E_{at} \) the atmospheric radiation (outdoors) [W/m²]

In general the coefficient \( h_{cv} \) depends on the air velocity near the surface and the difference between air and surface temperature. The air temperature near the surface can differ significantly from the temperatures further from surface.

The coefficient \( h_r \) and the mean radiant temperature depend on the temperatures, the emissivities and geometries of the surrounding surfaces, including the considered surface. The atmosphere is treated as a black body (emissivity=1) with the air temperature (as this is not true a correction \( \Delta E_{at} \) is included).

Above boundary condition can be written more concise by introducing a fictive temperature. This is only for convenience’s sake. We shall call this the effective temperature (\( T_{ev} \)). Sometimes this is called the sol-air temperature.

More concise: \( q_z = (h_{cv} + h_r)(T_{ev} - T_s) \)

Comparing this with the equation for the boundary condition leads to the expression for the effective temperature given below:

\[ T_{ev} = \frac{(h_{cv} \cdot T_a + h_r \cdot T_r + \sum_j \alpha_j E_j \cdot \varepsilon \Delta E_{at})}{(h_{cv} + h_r)} \]  

(1.11)

So the boundary condition is (the z-axis is perpendicular to the surface):
The sum of both heat transfer coefficients \( h_{cv} + h_{r} \) is called the combined surface heat transfer coefficient.

**Example**

Flat roof \( \alpha = 0.9 \), clear sky, summer, in daytime: \( h_{e} = 20 \text{ W/m}^2\text{K} \), Irradiance \( E_{sol} = 500 \text{ W/m}^2 \) and \( \Theta_{e} = 20^\circ\text{C} \)

So the effective temperature is: \( \theta_{rcv} = 20 + 0.9 \times 500/20 = 42.5^\circ\text{C} \)

Winter, at night: \( h_{e} = 20 \text{ W/m}^2 \), \( \varepsilon = 0.9 \), \( \Theta_{e} = 2^\circ\text{C} \), \( \Delta E_{at} = +100 \text{ W/m}^2 \) (extra atmospheric radiation loss during a clear sky).

\( \Theta_{rcv} = 2 - 0.9 \times 100/20 = -2.5^\circ\text{C} \).

With a characteristic length \( d \) the last boundary condition can be written as:

\[
-\lambda \left( \frac{\partial T}{\partial z} \right)_{s} = (h_{cv} + h_{r})(T_{rcv} - T_{s})
\]

(1.12)

The Biot number is a measure for the temperature difference in the material compared to the temperature difference between the surface and the ambient. For \( \text{Bi}<<1 \) the temperature distribution in the material is approximately uniform.

With a characteristic length \( d \) the last boundary condition can be written as:

\[
-\left( \frac{\partial T}{\partial z} \right)_{s} = \text{Bi}(T_{rcv} - T_{s}) \quad \text{and} \quad z' = z / d
\]

with Biot number \( \text{Bi} = d(h_{e} + h_{cv})/\lambda \). [-]

If \( \text{Bi} \rightarrow \infty \) it is called a Dirichlet boundary condition (1st kind); then the surface temperature is known. If the density of heat flow rate is known it is called a Neumann boundary condition (2nd kind) with as a special case for \( \text{Bi}=0 \) an adiabatic boundary condition (perfect thermal insulation). For all other values of the Biot number it is called a mixed condition (3rd kind).

The Biot number is a measure for the temperature difference in the material compared to the temperature difference between the surface and the ambient. For \( \text{Bi}<<1 \) the temperature distribution in the material is approximately uniform.

Calculation values of combined surface heat coefficient:

- Interior: \( h_{i} = 7 \text{ W/m}^2\text{K} \)
- Exterior: \( h_{e} = 25 \text{ W/m}^2\text{K} \)
1.3. Moisture transfer in materials

1.3.1 The moisture potential

The potential for heat transfer was obvious. Heat storage and heat conduction depend on the temperature and the temperature distribution in the material. For the moisture transfer it is less obvious. In order to understand this we shall start with a short summary of properties of humid air and moist material.

In building physics the state of a unity of mass depends usually on two physical quantities: the temperature and the pressure (or partial pressures). If this mass is in equilibrium with its surroundings (neither heat nor mass transfer) then all other quantities will depend on these ones (e.g. mass density and internal energy).

For ideal gasses the next equation applies:

\[ p = \frac{\rho}{R} \frac{RT}{M_g} \]

where:
- \( p \): the pressure [Pa]
- \( \rho \): the density [kg/m³]
- \( R \): universal gas constant (8.314 J/molK)
- \( T \): the absolute (=thermodynamic) temperature (273.15+\( \Theta \))
- \( M_g \): the molar weight (kg/mol)

Ideal gasses are gasses for which above relation applies. At the ruling pressures and temperatures in building physics air and vapour can be considered as ideal gasses. It is convenient to use instead of \( R/m_g \) the symbols \( R_a \) (= 287.1 J/kgK) for air and \( R_v \) (= 462 J/kgK) for vapour. So:

\[ p_a = \rho_a R_a T \quad (\approx 10^5 \text{ Pa}) \quad \text{en} \quad p_v = \rho_v R_v T \]

Here \( x \) is the humidity ratio [kg vapour/kg dry air] of humid air. Because \( p_a \approx 10^5 \) Pa (1 atmosphere) the relation between the vapour pressure and the humidity ratio can be derived from eq.1.13:

\[ x \approx \frac{p_v}{p_a} \frac{R_v T}{R_a T} \approx 0.62 \cdot 10^{-5} p_v \]

The vapour pressure and the humidity ratio are so-called absolute humidity’s.

The concentration of water vapour in air is: \( \rho_v = \rho_x \approx 1.2 x \). We shall not use this property but the humidity ratio instead.

The state in which the fluid phase and the gas phase are in equilibrium (e.g. no net evaporation nor condensation) depends on the temperature and pressure. So at 1 atmosphere (10⁵ Pa) ambient pressure there is for every temperature a maximum vapour pressure; the saturation pressure \( p_{sat}=p_{sat}(\Theta) \). This pressure cannot be exceeded as condensation prevents that. Obviously the pressure can be lower if there is no water available to evaporate. The saturation pressure is higher at a higher temperature.
Boiling occurs if the vapour pressure in the bubbles in the water equals the total ambient pressure (in the case above the air pressure) in order to be able to reach the surface instead of collapsing. At 1 atmosphere water will boil at 100°C, i.e. the saturation pressure at 100°C equals 10^5 Pa. Hence at an ambient pressure higher than 1 atmosphere water will only boil above 100°C (high-pressure-pan) and at lower pressures (e.g. high above see level) water boils below 100°C. At 0°C water is in equilibrium with its solid phase (ice). There is a (very low) pressure and temperature at which the three phases, solid, liquid and gas are in equilibrium: the Tripelpoint (0.01°C and 600Pa).

The relative humidity is defined as

\[
\varphi = \frac{p_v}{p_{sat}} = \frac{\rho_v}{\rho_{sat}} \text{ or } RH = 100\% \varphi
\] (1.15)

So at every temperature there is a vapour saturation pressure \(p_{sat}\). If the temperature of humid air decreases then at a specific temperature (the dewpoint temperature \(\Theta_{dew}\)) saturation will be reached and condensation will occur. The dewpoint is a measure of the absolute humidity (\(p_v = p_{sat}(\Theta_{dew})\)). Condensation occurs when the surface temperature in a room drops below the dewpoint.

For evaporation heat is needed and at condensation heat is released. If in a porous material this phase change occurs there will be a heat source (negative source in case of evaporation) in the pores. This source is often very small and negligible.

A wet surface will cool down as a result of evaporation. The final surface temperature will depend on the vapour pressure of the air near the surface, the saturation pressure of the wet surface and the heat that is transferred to that surface. If solely heat is transferred to the surface by convection (no conduction and no radiation) a minimum surface temperature arises: the so-called wet-bulb temperature \(\Theta_{wet}\). If there is a high air velocity near the wet surface of a thermal insulator the surface temperature will be close to the wet-bulb temperature.

Approximation for \(p_{sat}\)

\[
p_{sat} = 611 \exp \left( \frac{17.08\Theta}{234.18 + \Theta} \right) \text{ [Pa]}
\]

\(\Theta < 0°C \quad p_{sat} = 611 \exp \left( \frac{22.44\Theta}{172.44 + \Theta} \right) \text{ [Pa]}

An approximation for wet bulb is the solution of the equation:

\[p_{sat}(\Theta_{wet}) - \varphi p_{sat}(\Theta_a) = 66.71(\Theta_a - \Theta_{wet})\]

The approximate solution is:

\[\Theta_{wet} = \sqrt{(10.11 + 22.06\Theta_a)(1 - \varphi) + \varphi(\Theta_a + 14.85)^2} - 14.85 \text{ °C}\]

In the Mollier-diagramme the relations between vapour pressure, humidity ratio, relative humidity, dewpoint temperature, wetbulb...
temperature, enthalpy of moist air and the saturation pressure are all plotted in one figure.

Porous materials contain much more moisture than the quantity in the pore air. In pores water vapour turns out to condense already below the saturation pressure. The quantity of this condensate depends on the relative humidity in the pores and pore geometry (Chapter 4). The graph of the moisture content \( w \) (kg moisture/m\(^3\)) as a function of relative humidity is called the sorption curve or sorption isotherm (see figure).

Below a relative humidity of app. 98% moisture transfer is roughly dominated by molecular diffusion (the potential is the vapour pressure). Above 98% the moisture content increases enormously and moisture flow is caused by capillary suction (see Chapter 4). In thermodynamic equilibrium the suction \( p_c \) [Pa] is related to the relative humidity by Kelvin’s law:

\[
p_c = \rho_w \cdot R_v \cdot T \ln(\varphi)
\]

where:
- \( \rho_w \) = density of water (= 1000 kg/m\(^3\))
- \( R_v \) = gas constant of water vapour (= 462 J/kgK)
- \( T \) = absolute temperature

Instead of a plot of moisture content vs. relative humidity a plot of the moisture content vs. the suction is more convenient (the moisture retention curve) and depicts also directly the results of measurements.

So for moisture transfer equation there are 3 different possibilities for potentials: vapour pressure+temperature, relative humidity+temperature and suction+temperature. Besides this also the moisture content+temperature can be used.

### 1.3.2. The moisture transfer equation

In a porous material water vapour is above all transferred by molecular diffusion. The density of mass flow rate (mass flow per m\(^2\) material) is proportional to the vapour pressure drop per m material. Fick’s law applied to a porous material is:

\[
g = - \mu \frac{\delta}{\mu} \text{grad } p_c
\]

where:
- \( \delta_a \) = the vapour permeability of stagnant air [kg/smPa] or [s]
- \( \mu \) = the vapour diffusion resistance factor (always >1) [-]

For stagnant air \( \mu = 1 \) and for vapour retarders a very large number. Materials with a high open porosity \( \psi_o \) [m\(^3\) pore volume/ m\(^3\) porous material] usually have a low \( \mu \)-value. The diffusion resistance factor depends not only on the pore properties of the material (porosity, tortuosity) but also on the moisture content (see Chapter 4).

In a porous material below a relative humidity of \( \varphi < 0.98 \) the change of moisture content is usually written as:
Here $\xi$ [kg/m$^3$] is the specific hygroscopic moisture capacity (tangent of sorption isotherm).

After substitution of eq.1.17 and eq.1.18 in the mass balance (eq.1.5) the equation for vapour transfer yields:

$$\text{div} \frac{\delta_a}{\mu} \text{grad} \ p_v = \xi \frac{\partial p_v}{\partial t}$$  \hspace{1cm} (1.19)

If the material is very wet water is transferred in the material by capillary suction. Darcy’s law has the form of the Fourier’s law and Fick’s law; it relates the density of mass flow rate to the gradient of the capillary suction and the gravity:

$$\bar{g} = -k_m \text{grad} (p_c + \rho_w g z)$$  \hspace{1cm} (1.20)

where: $k_m$ = moisture permeability [kg/smPa] or [s]  
$z$ = vertical height  
$g$ = acceleration of free fall (gravitation constant) [m/s$^2$]  
$\rho_w$ = density of water  

The moisture permeability is a function of the moisture content. It is larger at a larger moisture content.

The storage term can be determined with the tangent of the moisture retention curve (section 1.3.1):

$$\frac{\partial w}{\partial t} = \frac{dw}{dt} \frac{\partial p_c}{\partial p_c} = \Xi \frac{\partial p_c}{\partial t}$$  \hspace{1cm} (1.21)

where: $\Xi$ = the specific capillary moisture capacity

So the equation for capillary transport (without a source term and gravity) is with eq.1.21 and the mass balance (eq.1.5):

$$\text{div} k_m \text{grad} p_c = \Xi \frac{\partial p_c}{\partial t}$$  \hspace{1cm} (1.22)

Because according to Kelvin’s law (eq.1.16) the capillary suction is a function of the relative humidity, eq.1.22 can also be written with the relative humidity as potential. (Section 1.3.1). By combining eq.1.19 and eq.1.22 a more general equation for water and vapour can be derived (Chapter 4)

Remarks
- Not all porous materials need to be capillar, e.g. mineral wool is not capillary i.e. does not transport water by suction.
- The equations for heat and moisture transfer in porous media are often called the Phillip-De Vries equations.
1.3.3. The boundary conditions

Analogous to the heat problem the density of moisture flow rate perpendicular to an interface is continuous (eq.1.8). Also the relative humidity and the capillary suction are continuous (eq.1.9).

So: \( \phi_1 = \phi_2 \) or \( p_{c1} = p_{c2} \) or \( g_{sl} = g_{s2} \)

For water flow the contact at the interface cannot always be regarded as ideal. In that case a contact moisture resistance is supposed, or

\( p_{c1} = p_{c2} + g_{sl} Z_c \)

where \( Z_c \) = the assumed contact moisture resistance

\( g_{sl} \) = the density of water flow rate normal to the interface

At the surface bordering the air the boundary condition can be a known relative humidity (Dirichlet boundary condition, eq.1.10).

In the same way as for heat transfer the density of moisture flow rate can be written as (z-axis perpendicular to the surface):

\[
\begin{align*}
\left( g_z \right)_s &= g_o + \beta (p_{vi} - p_s) \\
\left( g_z \right)_s &= \frac{\delta_a}{\mu} \frac{\partial}{\partial z} p_v \quad \text{of} \quad \left( g_z \right)_s = -k_m \frac{\partial}{\partial z} p_c
\end{align*}
\]

where: \( \beta = \) surface coefficient of water vapour transfer (s/m)

\( g_o \) = density of vapour flow rate perpendicular to the surface

\( g_o \) = extra density of moisture flow rate e.g. rain

The surface coefficients of vapour transfer and convective heat transfer are correlated. A formula is:

\[
\beta_x = (Le)^{n-1} \frac{h_{cv}}{c_p} \approx \left(0.8\right)^{n-1} \cdot 10^{-3} h_{cv}
\]

where: \( Le = \) Lewisnumber [-]

\( c_p = \) specific heat capacity of dry air (=1000J/kgK)

\( n = 0 \) for laminar and \( n = 1 \) for turbulent flow

and \( (g_o)_s = \beta_x (x - x_{sat}) \), \( x = \)humidity ratio

So with eq.1.14: \( \beta \approx \left(0.8\right)^{n-1} \cdot 0.62 \cdot 10^{-8} h_{cv} \)

Assumed is that the surface is completely wet. So often \( \beta \) will be smaller

1.3.4. Air flow in porous materials

In porous materials air is transferred if there are differences in the total pressure. The density of mass flow rate is analogous to eq.1.20:

\[
\bar{g}_a = -k_a \left( \nabla p + \rho_a \bar{g} \right)
\]

where: \( k_a = \) the air permeability [s]
\[ \mathbf{c}_z = \text{vertical unit vector [-]} \]

\[ p = \text{pressure} \]

\[ \rho_a = \text{density of the air in the material} \]

If the air is considered as incompressible and gravity is neglected then inserting eq.1.25 into the mass balance eq.1.5 yields:

\[ \text{div} \mathbf{g}_a = 0 \quad \text{or} \quad \text{div} k_a \text{grad} p = 0 \quad (1.26) \]

With known boundary conditions for the pressure this equation can be solved. Air transport through a material causes convective heat transfer and vapour transfer. Hence the equations for the density of heat flow rate (eq.1.6) and vapour transfer (eq.1.19) have to be extended with a convective term (\( c_p = \text{specific heat capacity of air} \)):

\[ \dot{q} = -\lambda \text{grad} T + c_p \mathbf{g}_a (T - T_{\text{ref}}) \]

and

\[ \rho c \frac{\partial T}{\partial t} = \text{div} \lambda \text{grad} T - c_p \mathbf{g}_a \cdot \text{grad} T \quad (1.27) \]

Where use is made of the notation of the dot product:

\[ \mathbf{g} \cdot \text{grad} T = g_x \frac{\partial}{\partial x} T + g_y \frac{\partial}{\partial y} T + g_z \frac{\partial}{\partial z} T \]

For example mineral wool can be very permeable and so this transfer can be important. In the same way the vapour transfer equation can be extended:

\[ \dot{\varphi} = -\frac{\delta_v}{\mu} \text{grad} p_v + \mathbf{g}_a x \quad (1.28) \]

and

\[ \xi \frac{\partial \varphi}{\partial t} = \text{div} \frac{\delta_v}{\mu} \text{grad} p_v - 0.62 \cdot 10^{-5} \mathbf{g}_a \cdot \text{grad} p_v \]

Use is made of eq.1.14 that gives the relation between the humidity ratio and the vapour pressure and eq 1.26.

In porous materials the convective heat transfer is often very small but the convective vapour transfer can be very significant.
If in eq. 1.25 the sum of the atmospheric pressure \( p_{\text{atm}} \) and \( \rho_a g z \) (=constant) is subtracted from \( p \) then with \( \rho_{a,m} (=\text{the air density in the material}) \) and \( \Delta p = p - p_{\text{atm}} \), eq. 1.25 can be written as:

\[
\tilde{g}_a = -k_a \left( \text{grad} \Delta p + (\rho_{a,m} - \rho_a)g \tilde{c}_z \right)
\]

The gravity term is \((\rho_{a,m} - \rho_a)g = \rho_a g (\rho_{a,m}/\rho_a - 1)\)

This quantity hardly changes by pressure variations e.g. in building physics 100 Pa is a very big change but compared to the atmospheric pressure only 1\%. The main cause of the air flows are temperature differences e.g. 27K is compared to the absolute temperature already 10\%.

Using the ideal gas law (eq. 1.13) and the assumption of a constant pressure yields:

\[(\rho_{a,m} - \rho_a)g = \rho_a g (T_a/T_m - 1) \approx -3.5 \cdot 10^{-3} \rho_a g (T_m - T_a)\]

This approximation is known as the Boussinesq approximation. The flow in the material resulting from temperature differences is called buoyancy.

1.3.5. Summary

<table>
<thead>
<tr>
<th>Heat</th>
<th>Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Phi_m + \Phi_p = \Phi_\text{out} + \frac{d}{dt} Q_{\text{acc}} )</td>
<td>( G_m + G_p = G_\text{out} + \frac{dm}{dt} )</td>
</tr>
<tr>
<td>(- \text{div} \tilde{q} + S_e = \frac{\partial}{\partial t} e)</td>
<td>(- \text{div} \tilde{g} + S_m = \frac{\partial w}{\partial t})</td>
</tr>
<tr>
<td>( \tilde{q} = -\lambda \text{grad} T )</td>
<td>( \tilde{g}_v = -\frac{\delta}{\mu} \text{grad} p_v )</td>
</tr>
<tr>
<td>( \tilde{g}_t = -k_m \text{grad} (p_c + \rho_w g z) )</td>
<td>( \tilde{g}_t = -k_m \text{grad} (p_c + \rho_w g z) )</td>
</tr>
<tr>
<td>( \rho c \frac{\partial T}{\partial t} = \text{div} \lambda \text{grad} T )</td>
<td>( \xi \frac{\partial p}{\partial t} = \text{div} \frac{\delta}{\mu} \text{grad} p_v )</td>
</tr>
<tr>
<td>( \Xi \frac{\partial p_c}{\partial t} = \text{div} k_m \text{grad} p_c )</td>
<td>( -\lambda \left( \frac{\partial T}{\partial n} \right)<em>s = (h</em>{\text{cv}} + h_v)(T_{\text{cv}} - T_s) )</td>
</tr>
<tr>
<td>( (g_x)<em>s = g_v + \beta (p_v - p</em>{v,s}) )</td>
<td>( (g_x)_s = \frac{\delta}{\mu} \frac{\partial}{\partial x} p_v )</td>
</tr>
<tr>
<td>( (g_t)_s = -k_m \frac{\partial}{\partial x} p_c )</td>
<td>( (g_t)_s = -k_m \frac{\partial}{\partial x} p_c )</td>
</tr>
<tr>
<td>( h_{\text{cv}} )</td>
<td>( \beta \approx (0.8)^{n-1} \cdot 0.62 \cdot 10^{-8} h_{\text{cv}} )</td>
</tr>
</tbody>
</table>
2. Steady State Heat Conduction

2.1. Introduction

In reality steady-state heat conduction will never happen. Exterior and interior temperature, solar radiation and wind are all time dependant. In fact steady state calculations provide but average heat flows and average temperatures of a long time interval so that storage effects can be neglected. Strictly speaking this interpretation is only true if thermal conductivities and heat transfer coefficients are constants i.e. if the equation is linear.

The equation of steady state heat conduction with a constant thermal conductivity is (see eq.1.7):

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0
\]  
(2.1)

If also the surface heat transfer coefficients are constant then (see eq.1.12):

\[
-\lambda \frac{\partial T}{\partial n} = (h_{cv} + h_t)(\bar{T}_{cv} - \bar{T}_n)
\]  
(2.2)

From now on the average-signs (bars) will be omitted.

Simple calculations are only possible for one-dimensional heat transport i.e. the temperature is dependant on one spatial co-ordinate (perpendicular to the surface). In practice this is an approximation as the effective temperature is usually different for every point on the structure. Moreover a structure seldom has a cross-section and composition that allows for a one-dimensional treatment. On its best with one-dimensional calculations the area averaged values of temperatures are found.

By the complexity and the complicated boundary conditions of a building detail it is generally not feasible to make manual calculations and the computer has to offer the solution. Important is:

- to be aware of the consequences of the assumptions and simplifications that are necessary for a computer simulation;
- to understand the results of the computer and to be able to judge them on likelihood
- to make estimations with manual calculations whenever possible.
Integrating the heat conduction equation (1.5) with constant thermal conductivities over a time interval yields:

\[
\frac{1}{\Delta t} [\rho c T(\Delta t) - \rho c T(0)] = \lambda \frac{\partial^2 T}{\partial x^2} + \lambda \frac{\partial^2 T}{\partial y^2} + \lambda \frac{\partial^2 T}{\partial z^2}
\]

If the interval is sufficient long the l.h.s. can be neglected and the steady state equation is obtained.

### 2.2. Isotherms and heat flow lines

An isothermal surface is a fictitious surface connecting all points in a structure on which the temperature has the same value.

If these surfaces are drawn for a range of temperatures with an equal interval e.g. 10°C, 12°C, 14°C, 16°C etc. a picture arises of the temperature distribution in the structure. On places with a small distance between the isothermal surfaces the gradients perpendicular to the surfaces are high (large \(\partial T/\partial n\)). That doesn’t imply that the density of the heat flow rate is also large as this density is also dependant on the thermal conductivity: \(q = -\lambda \frac{\partial T}{\partial n}\).

The direction of the density of heat flow rate is perpendicular to the isothermal surface as the gradients in the tangent surface are zero.

The line of which the tangent line in every arbitrary point of it is the direction of the heat flow is called a heat flow line or adiabatic.

This line has nothing to do with the magnitude of the density of heat flow rate but exclusively with the direction (compare this with streamlines of a laminar fluid flow).

In the same way as for fluids the heat flow lines can form a tube. Within this tube the total heat flow is constant as there is no flow perpendicular to the envelope of the tube (energy conservation). At a narrowing of the tube the density of heat flow rate is larger as the cross-section is smaller (a rapids for fluid flow). If this happens in a homogeneous material the isothermal surfaces will be more closely to each other.

Drawing isothermal surfaces and heat flow lines in a 3-dimensional body is very complicated. Usually this representation of the temperature and vector heat flow field is given for cross-sections and surfaces. Then the isothermal surfaces are isothermal lines: the isotherms.

For a 2-D problem (no heat flow perpendicular to the cross-section) these lines give a very illustrative picture of the temperature field. The heat flow lines are drawn in such a way that between two successive lines the heat flow rate per meter perpendicular on the drawing (the linear density of heat flow rate) has always the same magnitude. This is
called the increment. The region between two heat flow lines is called a lane. The total linear density of heat flow rate through a certain section is simply the number of lanes multiplied by the increment.

If in a (part of the) structure the heat transfer is one-dimensional the isotherms will be straight parallel lines. The same goes for the heat flow lines. Furthermore the spacing between the heat flow lines is equal as the density of heat flow rate is the same everywhere in that part of the structure.

In insulation layers the distance between isotherms is very small (large gradient).

2.3. Thermal bridges

A thermal bridge is a part of the structure where the local density of heat flow rate at the inside surface is significantly larger than at other parts on the surrounding surface.

For a positive temperature difference between indoor and outdoor air this larger density of heat flow rate implies a lower surface temperature at the thermal bridge. Although the size of the thermal bridge can be small compared to the surface area of the rest of the structure the influence can be significant.

The thermal bridge disturbs the temperature field of the structure: the isothermal surfaces are not parallel to the surface any more and the heat flow lines are not straight and parallel to each other.

A thermal bridge entails two important drawbacks:

a. An increased local density of heat flow rate: extra heat loss of a structure

In the past the extra heat loss of buildings by thermal bridges was negligible compared to their very large total heat loss. This is not true any more and the loss by thermal bridges compared to the lower total loss nowadays can be significant.
b. A low surface temperature at the interior side of the thermal bridge when the exterior temperature is low.

A low surface temperature of a thermal bridge involves often a high surface relative humidity. This can result in moulds, rot and especially by condensation colouring of the surface.

Except these two effects there are some more drawbacks that will not be treated in this chapter:

c. A low surface temperature can have a negative influence on thermal comfort.

Especially relative large thermal bridges like columns in the facade, can cause a heat flow rate from a person to the cold surface by radiation (asymmetric radiation)

d. Interstitial condensation can occur.

The inside of a structure in the vicinity of the thermal bridge gets colder in winter and increase the risk for interstitial condensation. If there is insufficient time for drying, moisture can accumulate. If the drying time is sufficient long, particles from the inside of the structure can migrate with the vapour to the surface and cause a dirty spot (sometimes a salt eruption). This moisture can enhance corrosion, harm the solidity of building materials and cause frost damage. Also coatings on thermal bridges will flake off and plaster will come loose.

e. Cracks can appear and at a fluctuating thermal climate (e.g. irradiance by solar radiation) an annoying noise can be heard.

As the temperature near the thermal bridge differs from the temperature of the rest of the structure the thermal expansion differs as well. This strain will result into stresses and cause cracking.

Thermal bridges usually have a complicated geometrical shape. In practice calculations on such a thermal bridge by means of solving analytically the multi-dimensional steady state equation with boundary conditions is impossible. Nowadays very user-friendly software is available to deal with these problems. The use of such a black box enhances the importance of having insight in the problem: e.g. where large temperature gradients are to be expected (considering the discretisation grid), how much of the structure around the thermal bridge has to be part of the simulation (where are symmetry planes or symmetry axis), what result can be expected (computer result can be very wrong by an erroneous input)? Sometimes manual calculations can be useful as a first estimate.

It can happen that the computer is superfluous as the manual calculation is just as accurate and offers more insight in the important parameters. In the past building codes and standards were exclusively based on manual calculations. At present this is not true anymore.
2.3.1. Thermal bridges in plane structures

There are thermal bridges in plane structures if:

a. There is a local interruption of a layer in the structure. Usually this concerns an interruption of the thermal insulation layer.

b. Two dissimilar structures join e.g. window frame and facade

Often a distinction is made between linear and concentrated thermal bridges. Linear thermal bridges are narrow compared to the width of the structure and they are long compared to the thickness of the structure. If the surface area of the structure is much larger than the surface area of the thermal bridge this bridge is referred to as a concentrated thermal bridge.

For a linear thermal bridge the isotherm of the lowest interior surface temperature is a straight line. For a concentrated or point-shaped thermal bridge this isotherm is a closed line (e.g. a circle).

Near a linear thermal bridge the heat flow is 2-dimensional (2D). In the neighbourhood of concentrated thermal bridges the heat flow is 3-dimensional (3D) or in case of cylinder symmetry 2-dimensional (no heat flow in the tangent direction). The extra heat loss of a linear thermal bridge can be defined with the linear thermal transmittance $U_l$ and of a concentrated thermal bridge with a heat loss coefficient $L$:

$$\Phi = \ell U_l (T_i - T_e) \quad \text{and} \quad \Phi = L(T_i - T_e)$$

2.3.2. Thermal bridges at corners between plane structures

Usually a corner has three planes (three-plane corner) or two planes. In the latter case the problem is 2-dimensional. Examples are:

- corner between facade and interior partition structure (floor, wall);
- junction of facade and roof;
- corners between perpendicular adjacent facades.

If the facades in the last example are identical and the thermal insulation is not interrupted the thermal bridge is purely the result of the geometry.

The surface temperature at an inside corner is lower than could be expected from the thermal bridge effect. This phenomenon has several reasons:

- The air velocity in the corner is lower than elsewhere near the vertical wall and by that the surface coefficient of convective heat transfer is smaller
- There is radiation exchange between the surfaces close to the corner and consequently these surfaces exchange less heat with the room. Out of the corner this effect becomes smaller.
In the inside corner between a horizontal structure and a vertical also the temperature stratification of the room air will play a part. As a rule the temperature near the ceiling is higher than near the floor.

2.3.3. Constructions interrupting the insulation.

Fins and cantilevers can interrupt the insulation of the facade e.g. a balcony floor that is part of the floor in the building. In winter this slab acts as a cooling fin and the inside surface temperature will be low. In summer when much solar radiation is absorbed by this slab heat can be transferred to the interior and contribute to the cooling load. Laying a prefabricated balcony slab on consoles or on sidewalls can prevent this thermal bridge.

Another example of such thermal bridge is at the junction of the facade and the foundation. Applying insulation is difficult by the high static load on this junction.

2.4. One-dimensional steady-state heat conduction

One-dimensional steady-state heat conduction means that there is only one independent variable: e.g. if the temperature in a plane structure depends only on the distance to one of the surfaces. In this case in each homogeneous layer the temperature varies linearly with this distance and the density of heat flow rate has everywhere the same magnitude.

The temperature difference between two sides of a layer with a thickness $d$ is:

$$ T(0) - T(d) = q \frac{d}{\lambda} = qR $$

where $R$ is the well-known thermal resistance $[\text{m}^2\text{K}/\text{W}]$. Addition of the temperature differences across all layers (n layers) including boundary layers at the interior and exterior included, yields:

$$ T_i - T_e = q \left( \sum_{k=1}^n R_k + R_i + R_e \right) = \frac{q}{U} \quad (2.3) $$

with $U =$ total thermal transmittance

The temperature at the surface of each layer is:

$$ T_j = T_e + \left( R_e + \sum_{k=1}^{j-1} R_k \right) q \quad (2.4) $$

So the surface temperature is:

$$ T_s = T_e + \frac{(T_i - T_e)(R_e + \sum_{k=1}^n R_k)}{(R_i + R_e + \sum_{k=1}^n R_k)} \quad (2.5) $$

The ratio $(T_s - T_e)/(T_i - T_e)$ for this structure is independent of the temperatures and only dependant on the composition of the structure and the value of the surface coefficients.
One-dimensional heat conduction in a circular geometry means that the temperature depends only on the distance from the centreline (radial distance). For example in a good approximation the temperature in the insulation around a heating pipe depends solely on the radial distance. In this case the temperature varies logarithmic with this distance instead of linear as in the case of a plane wall.

The density of heat flow rate is:

\[ q = -\lambda \frac{\partial T}{\partial r} = \frac{T_1 - T_2}{\ln r_2 / r_1} \frac{1}{r} \]

Where \( r \) is the radial distance, \( r_1 \) the radius of the circle formed by the internal surface of the pipe and \( r_2 \) the external side.

The density of heat flow rate is not constant, but depends on \( r \). The total heat flow rate must be constant however (in = out). If the pipe has length \( \ell \) then the heat flow rate is:

\[ \Phi = q \cdot A = \frac{2\pi \cdot \ell}{\pi} \frac{T_1 - T_2}{\ln r_2 / r_1} \]

In the radial direction the density of heat flow rate depends on the radial distance and has not a fixed value as was true for a plane structure. So the thermal resistance is not constant. For cylindrical systems it is more convenient to use the linear thermal resistance \( R_\ell \) defined by:

\[ \Phi = \frac{\ell}{R_\ell} (T_1 - T_2) \quad \text{instead of} \quad \Phi = \frac{A}{R} (T_1 - T_2) \]

So:

\[ R_\ell = (\ln r_2 / r_1) / 2\pi\lambda \quad (\text{mK} / \text{W}) \]

The linear thermal resistance has a particular value for each layer. For example if there are \( n \) layers around the pipe with a different thermal conductivity and the combined surface coefficient at the external surface is \( R_i \) then the total thermal resistance from internal surface to the external environment is:

\[ R_i = \sum_{k=1}^{n-1} \left( \ln \frac{r_{k+1}}{r_k} \right) / 2\pi\lambda_k + R_i / 2\pi r_n \quad (2.6) \]

and

\[ \Phi = \ell \Delta T / R_i = \ell U_i \Delta T \]

where: \( \Delta T \) = the total temperature difference.

\( U_i = \) the linear U-value
The one-dimensional equation of a homogeneous layer of a plane structure is:

\[ \frac{\partial^2 T}{\partial x^2} = 0 \]

The solution is: \( \frac{\partial T}{\partial x} = a \rightarrow T = ax + b \)

where \( a \) and \( b \) are constants which follow from the boundary conditions:

\[-\lambda \frac{\partial T}{\partial x} = q \quad \rightarrow \quad q = -\lambda a \]

\( x = 0 \) \( T = T(0) \rightarrow T = -q x/\lambda + T(0) \)

So the temperature decreases linearly with \( x \).

Thermal problems often can be presented very clearly in a graphical way by an equivalent thermal circuit. The unknown temperatures of a steady-state problem are the nodes of the circuit. The sum of all heat flow rates (in compliance with the sign) to a node (i) is zero (conservation of energy):

So \( \sum_m \Phi_{m \rightarrow i} = 0 \)

A resistance is defined as:

\[ R = A(T_i - T_i)/\Phi \]

Where \( A \) is the surface area related to \( R \).

So, if at a node a number of resistances \( R_{m} \) are connected and also a known heat flow rate \( \Phi_0 \) is absorbed (e.g. solar radiation) then:

\[ \sum_m \frac{T_m - T_i}{R_{im} / A_{im}} + \Phi_0 = 0 \]

The total resistance of resistances in series is the sum of all:

\[ R_v / A_v = \sum_i R_i / A_i \]

The total conductance (inverse resistance) of resistance’s in parallel is the sum of all conductances:

\[ \frac{A_v}{R_v} = \sum_i \frac{A_i}{R_i} \]

For a plane wall with one-dimensional heat transfer the surface area \( A \) is the same for all layers.
The equation for steady state heat conduction in cylinder coordinates is (circle symmetry)
\[
\frac{\partial}{\partial r} r \frac{\partial}{\partial r} T = 0
\]
where: \( r \) = distance to the centreline

The solution is:
\[
r \frac{\partial}{\partial r} T = a \quad \rightarrow \quad T = a \ln r + b
\]

Density of heat flow rate is
\[
-\lambda \frac{\partial T}{\partial r} = q \quad \rightarrow \quad q = -\frac{\lambda \cdot a}{r}
\]

For example the boundary conditions of an insulation layer around a steel pipe can be:
\[
r = r_1: \quad T = T_1 \quad \quad r = r_2: \quad T = T_2
\]
with: \( T_1 \) = temperature of the tube
\( T_2 \) = surface temperature of the insulation

With these boundary conditions the solution is:
\[
T = T_1 \cdot \frac{T_1 - T_2}{\ln(r_2 / r_1)} \ln(\frac{r}{r_1})
\]
(In = natural logarithm). So in this case the temperature decreases not linear but logarithmic with the distance from the centreline.

2.5. One-dimensional approximations

2.5.1. Sketching isotherms and heat flow lines
In order to sketch the isotherms and heat flow lines in a two-dimensional plane structure with thermal bridges one proceeds as follows:

a. **Determine the straight heat flow lines.**
In a plane homogeneous layered structure the heat flow lines are straight and perpendicular to the surface. If there is a thermal bridge in this structure this is only true for the heat flow lines far from this thermal bridge (e.g. 5x the thickness of the structure). In the direction perpendicular to lines of symmetry there is no heat flow (obvious consequence of the symmetry) and the heat flow line coincides with the symmetry line.

For instance, if in a plane structure there are equidistant columns, the lines of symmetry are on the midplane between two columns and the midplane of the columns. If there is only one column there is one symmetry line and there are straight heat flow lines far from the column.
For a picture of the heat flow lines and the isotherms it is sufficient to consider only a section between two straight heat flow lines (the calculation detail).

b. **Draw the isotherms far from the thermal bridge.**
Far from the thermal bridge the heat transfer is one-dimensional and the temperature distribution in the structure can be calculated in the usual way. The isotherms can be drawn parallel to the surface.

c. **Estimate the temperature distribution on the line of symmetry of the thermal bridge.**
This cannot be done accurately without a computer but an estimate is found by assuming one-dimensional heat transport in the thermal bridge. At the warm side of the structure the temperature at a certain depth will be lower than at the same depth far from the bridge. The opposite is valid for the cold side. So there are fewer isotherms in the thermal bridge than in the rest of the structure. By connecting the isotherms far from the thermal bridge with points with the same temperature on the line of symmetry and perpendicular to this line a picture of isotherms is obtained. One of the isotherms is a straight line and is called the neutral isotherm. If there is an insulation layer the neutral isotherm will be in this layer as most of the isotherms are there.

The position of the neutral isotherm can be determined by intersecting the temperature profile far from the thermal bridge and the profile calculated with the one-dimensional assumption in the bridge (see figure below). If there is a thick layer at the cold side with a high thermal conductivity computer calculations show that the neutral isotherm is closer to this cold surface as with this estimation. The reason is that the conducting layer takes the effect of a cooling fin and sucks much heat from the thermal bridge so that the temperature in the bridge is lower than estimated. In the opposite case the neutral isotherm is closer to the warm side.
d. Draw the heat flow lines perpendicular to the isotherms
Near the thermal bridge these heat flow lines will bend away from the bridge. As the heat flow through the thermal bridge is relatively large the distance between two lines is smallest near the line of symmetry of the bridge and the minimum distance is found on the neutral isotherm.

2.5.2. The equivalent thermal transmittance
For a plane structure with a thermal bridge an equivalent thermal transmittance can be defined: The total heat flow divided by the temperature drop over the structure per m$^2$ of surface area [W/m$^2$K]. This coefficient can be estimated by considering physically two extreme situations that give an upper and a lower limit to the possible value of this coefficient.

i. All heat flow lines are straight and perpendicular to the surface. So no lateral heat exchange is assumed as if ideal insulating thin layers are applied at right angles through the structure. In reality the temperature drop over a thermal bridge is larger by lateral heat flow resulting in a larger heat transfer. So the thermal transmittance found in this way is the lower limit.

ii. All isotherms are straight and run parallel with the surface. In this case the physical maximum possible lateral heat exchange is assumed as if ideal conducting thin layers parallel with the surface are applied. In reality the temperature drop over a thermal bridge is lower and the U-value found in this way is an upper limit.

In order to derive formulas for the minimum and maximum U-value a rectangular grid is placed over the structure and in such a way that each rectangle contains but one material. The side of a rectangle parallel to the surface (the section width) has index $k$ and the other side (layer thickness) has index $j$. Rectangle $(j,k)$ has a surface area $A_k$ (so $\sum A_k=A_{tot}$) and a thermal resistance $R_{jk}$.

In case (i) the total transmittance is the area weighed mean of the transmittances of each section (index $k$). The transmittance of a section is the inverse of the sum of resistances of its layers (index $j$):

$$U_{min} = \frac{1}{A} \sum_k \frac{A_k}{R_k + R_{e} + \sum_j \frac{1}{R_{kj}}}$$  \hspace{1cm} (2.7)

In case (ii) the total resistance is found as the sum of resistances of each layer (index $j$). The resistance of a layer is the inverse of the area weighed mean conductance of the sections (index $k$) of the layer.

$$U_{max} = \frac{1}{R_k + R_{e} + \sum_j \frac{A}{\sum_k \frac{A_k}{R_{kj}}}}$$  \hspace{1cm} (2.8)
So the equivalent thermal transmittance $U$ is between $U_{\text{min}}$ and $U_{\text{max}}$. An estimate is given by:

$$\frac{1}{U} = 0.5 \left( \frac{1}{U_{\text{min}}} + \frac{1}{U_{\text{max}}} \right)$$  \hspace{1cm} (2.9)

The insulation in a wooden floor is between the beams. Hence this structure can be divided into two sections. The structure has three layers: floor, insulation, and ceiling. Let $f_A = A_{\text{beam}}/A_{\text{tot}}$. The minimum thermal transmittance (case (i)) is:

$$(i) \quad U_{\text{min}} = \frac{f_A}{\sum R_{kj} + R_i + R_e} + \frac{1 - f_A}{\sum R_{cj} + R_i + R_e}$$

The maximum thermal transmittance (case (ii)) is:

$$U_{\text{max}} = \frac{1}{\sum_j 1/ \left( \frac{f_A}{R_{kj}} + \frac{1 - f_A}{R_{cj}} \right) + R_i + R_e}$$

In order to get insight in the consequences of the position of insulation on the heat transmittance and surface temperature we consider three different cases (see figure):

1. A concrete wall with inside insulation: height 1 m, total thickness: 0.2 m, \( \lambda_{\text{concrete}} = 2 \text{ W/mK} \), thickness of insulation 0.04 m, \( \lambda_{\text{ins}} = 0.02 \text{ W/mK} \). \( R_i + R_e = 0.17 \text{ m}^2\text{K/W} \). A linear thermal bridge of concrete interrupts the insulation: width 0.1 m.

2. The insulation is placed exactly in the middle of the wall with the same total thickness and the same interruption in the middle now.


For all 3 cases the minimum total thermal transmittance is:

\[
U_{\text{min}} = \frac{0.1}{0.2 + 0.17} + \frac{0.9}{0.16 + 0.04 + 0.17} = 0.77 \text{ W/m}^2\text{K}
\]

The maximum:

\[
U_{\text{max}} = \frac{1}{0.16 + \frac{1}{0.1} + 0.9 + 0.17} = 2.31 \text{ W/m}^2\text{K}
\]

So the average value (eq. 2.9) is for all cases:

\[
U = \frac{2 \cdot 0.77 \cdot 2.31}{0.77 + 2.31} = 1.155 \text{ W/m}^2\text{K}
\]
The results of computer simulations with $\Theta_e = 0$ and $\Theta_i = 10^\circ C$ are ($\Theta_{si}$ is the lowest surface temperature):

1. $U = 0.89 \, \text{W/m}^2\text{K}$ \hspace{1cm} $\Theta_{si} = 3.3^\circ C$
2. $U = 1.18 \, \text{W/m}^2\text{K}$ \hspace{1cm} $\Theta_{si} = 6.8^\circ C$
3. $U = 1.16 \, \text{W/m}^2\text{K}$ \hspace{1cm} $\Theta_{si} = 7.4^\circ C$

For a 0.2m thick massive wall without insulation the surface temperature is $\Theta_{si} = 5.2^\circ C$. The simplified method gives for all cases: $U=1.155 \, \text{W/m}^2\text{K}$

**Comments on this result**

The results are between $U_{\text{min}}$ and $U_{\text{max}}$. $U_{\text{max}}$ is clearly too pessimistic. The lateral influence of the thermal bridge is much smaller than 1m so that the straightening of the isotherms 1m leads to a too high maximum value.

Except for interior insulation (case 1) the simplified method gives reasonable results.

Dependent on the place of the insulation very different surface temperatures are obtained. With inside insulation the concrete wall is a kind of cooling fin and the minimum surface temperature is lower than for a massive wall without insulation! This is an example of a situation that moisture problems arise after insulation (surface condensation). Outside insulation has the opposite effect. The heat flow lines diverge at the massive inside wall with as a consequence a higher surface temperature. Note that there is a combination of a positive and a negative effect:
- interior insulation: low heat loss and moisture problem
- exterior insulation: high heat loss and no moisture problem

This is understandable: by extending the interior surface of a thermal bridge both the heat loss and the surface temperature increase.

**2.5.3. Thermal bridges of materials with a large conductivity**

The simplified method is an example of a one-dimensional approximation. Also thermal bridges with a large conductivity and no lateral heat flow can be treated as a one-dimensional problem. A high conductivity (e.g. metal) means that temperature gradient in the thermal bridge can be neglected (Biotnumber is small). No lateral heat flow means that the heat flow remains within the thermal bridge. Two examples will be given.

\[ \Phi_i = (T_i - T_e)/(R_i/A_i + R_e/A_e) \]

The linear thermal transmittance is:
The surface temperature is:

\[ T_{mi} = T_m \left( \frac{R_e/A_e}{R_i/A_i + R_e/A_e} \right) \left( T_i - T_e \right) + T_e \]

The property \( A/R \) is called the ‘thermal permeance’.

The question is what is more advantageous with a fixed value of \( A = A_i + A_e \), the frame as far as possible to the interior or to the exterior?

As \( R_i > R_e \) a large inside surface will result in the largest heat loss but also the highest surface temperature. The latter is usually most important. Sometimes this is a reason to enlarge the inside surface area on purpose.

**b. Linear thermal bridge formed by a metal frame with a thermal break (an interruption of insulating material).**

The heat flow is:

\[ \Phi = (T_i - T_e) / \left( R_i/A_i \right) \left( R_e/A_e + d/\lambda_{is} A_{is} + R_e/A_e \right) \]

where:  
- \( d \) = the thickness of the thermal break  
- \( A_{is} \) = the cross-section  
- \( \lambda_{is} \) = the thermal conductivity

The interior surface temperature is:

\[ T_{mi} = \left( 1 - R_i/A_i \right) / \left( R_i/A_i + d/\left( \lambda_{is} A_{is} \right) + R_e/A_e \right) \left( T_i - T_e \right) + T_e \]

\[ R_i = 1/8, \quad R_e = 1/25, \quad A_e = 0.103 \times \text{length}, \quad A_i = 0.129 \times \text{length}, \quad \lambda_{is} = 0.2 \text{ W/mK}, \quad A_{is} = 0.017 \times \text{length}, \quad d_{is} = 0.003 \text{ m}, \]

\[ \Phi = \frac{\text{length} h \cdot (T_i - T_e)}{1 + 0.032 + 0.135} \]

\[ = 0.45 \cdot \text{length} h \cdot (T_i - T_e) \]

If e.g. the total thermal transmittance of the glazing is 3W/m²K, this heat loss is equivalent to the loss of a strip of glazing with a width of 0.45/3m=0.15m

\[ T_{mi} = \left( \frac{0.45}{8 \cdot 0.129} \right) \cdot (T_i - T_e) + T_e = 0.56 \cdot (T_i - T_e) + T_e \]

The surface temperature is very low. Problems with condensation on this frame are to be expected.
Assumption: the temperature of the metal is homogeneous ($T_m$)

The heat flow rate to the frame without thermal break is:

$$\Phi_i = (T_i - T_m) \frac{A_i}{R_i}$$

The heat flow rate from the frame to the exterior is:

$$\Phi_e = (T_m - T_e) \frac{A_e}{R_e}$$

As lateral heat exchange can be neglected $\Phi_i = \Phi_e$. That yields $T_m$ and $\Phi_i$ (the equation below a)

The heat flow rate in the frame with a thermal break is:

$$\Phi = (T_i - T_{mi}) \frac{A_i}{R_i} = (T_{mi} - T_{me}) \frac{A_m}{d} \frac{A_e}{d} \frac{A_i}{R_i} = (T_{me} - T_e) \frac{A_e}{R_e}$$

Also here there is a connection in series of ‘thermal resistances’:

$$\frac{1}{R_i} = \frac{1}{d} \frac{1}{\lambda} + \frac{1}{A_i} + \frac{1}{A_e} + \frac{1}{A_m} / d$$

This yields the equation below b.

### 2.6. The temperature ratio

#### 2.6.1. The surface temperature

In the one-dimensional situation the inside surface temperature ($T_s$) can be calculated if the interior and exterior temperature and the ratio of thermal resistances are known, viz.:

$$T_s = \frac{R_i}{R_i + R_m} T_e + \frac{R_m}{R_i + R_m} T_i = \frac{R_i}{R_i + R_m} T_e + \frac{R_m}{R_i + R_m} (T_i - T_e)$$

where: $R_m =$ total thermal resistance between the surface and the exterior air ($T_e$)  
$R_i =$ surface heat transfer resistance at the inside ($T_i$)

So the surface temperature depends on the thermal environment at both sides and a factor that depends on the structure. This factor was the starting point for the introduction of the surface temperature factor. In the more general (3-dimensional) case this factor is however less easy to calculate.

Let a structure be bounded by n surfaces with n different bordering temperatures $T_{e1}$, $T_{e2}$, ..., $T_{en}$. For example the boundary conditions are (1.10a):

$$1 \leq k \leq n, \quad -\lambda_k \left( \frac{\partial T}{\partial n} \right)_k = h_{ek} (T_{ek} - T_{ek})$$

Suppose $a_k(x, y, z) T_{ek}$ is the solution of the differential equations where, except for surface k, all bordering temperatures equal zero. From the
linearity (see end of section) of the equations with boundary conditions then follows that \( a_k(x, y, z) \) is a solution of the equations with: \( T_{ek} = 1 \) and for \( i \neq k \): \( T_{ei} = 0 \). If we do this successively for all surfaces the general solution is:

\[
T(x, y, z) = \sum_{k=1}^{n} a_k(x, y, z) T_{ek}
\] (2.10)

The functions \( a_k(x, y, z) \) are independent of the bordering temperatures. If \( T_{ek} = T_e \) for \( 1 \leq i \leq n \) then also: \( T(x, y, z) = T_e \), so:

\[
\sum_{k=1}^{n} a_k(x, y, z) = 1
\]

If the value of \( a_i(x, y, z) \) at the surface is denoted as \( a_{sk} \) and the inside surface has index 1 with surface temperature \( T_s \) and \( T_{e1} = T_1 \) then the inside surface temperature is:

\[
T_s = \sum_{k=1}^{n} a_{sk} T_{ek} = a_{s1} T_1 + \sum_{k=2}^{n} a_{sk} T_{ek}
\] (2.11)

So \( T_s \) is a weighed mean of the bordering temperatures. The weighing only depends on the position on the inside surface and is independent of the bordering temperatures. If there are \( n \) different bordering temperatures then with \( n-1 \) different temperature-independent functions a general solution can be found.

\[
T_s = a_2 T_{e2} + a_1 T_{e1} + a_3 T_{e3}
\]

The functions \( a_i(x, y, z) \) can also be used to calculate the heat loss of the structure:

\[
\Phi = h_i A \sum_{k=2}^{n} \overline{a_{sk}} (T_i - T_{ek})
\]

where: \( \overline{a_{sk}} \) = the area weighed mean of \( a_{sk} (x, y, z) \)
\( A \) = total inside surface area
\( h_i \) = surface heat transfer coefficient
\[ \Phi = \int h_i (T_i - T_n) \, dA = h_i \, A \left( \overline{T_i} - \sum_{k=1}^{n} a_k T_{ek} \right) \]

where: \[ \int dA = \text{the surface integral over the inside surface. Because} \]
\[ \sum_{k=1}^{n} a_k = 1 \text{ and } T_{e1} = T_i \text{ this is equal to the equation above.} \]

The horizontal cross-section of the partition wall between a living room and a garage is given schematically in the figure above. The garage is badly insulated and only moderately heated. The thickness of the insulation material on the 20cm thick concrete wall amounts to 4cm. In the table below the mean of measured values during two nights are given: the air temperatures as well as the surface temperature \( \Theta_x \) at the top corner of the living room.

<table>
<thead>
<tr>
<th>Living room</th>
<th>garage</th>
<th>exterior</th>
<th>( \Theta_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20°C</td>
<td>5°C</td>
<td>-10°C</td>
<td>13.1°C</td>
</tr>
<tr>
<td>20°C</td>
<td>5°C</td>
<td>-5°C</td>
<td>13.5°C</td>
</tr>
</tbody>
</table>

Calculate the surface temperature for the following temperatures:
- living room: 20°C, garage 10°C and \( \Theta_e = 0°C \).

Solution:

\[
\begin{align*}
13.1 &= -10 \, a_1 + 5 \, a_2 + 20 \, a_3 \\
13.5 &= -5 \, a_1 + 5 \, a_2 + 20 \, a_3 \\
1 &= \, a_1 + a_2 + a_3
\end{align*}
\]

This yields: \( a_1 = 0.08 \); \( a_2 = 0.3 \); \( a_3 = 0.62 \)

So the temperature \( \Theta_x \) is:

\[ \Theta_x = 0 \cdot 0.08 + 10 \cdot 0.3 + 20 \cdot 0.62 = 15.4°C \]
If the heat conduction and the surface coefficient of heat transfer are constant (independent of the temperature or the spatial coordinates) the problem is linear. Linear problems are simpler to solve than non-linear ones. If there are \( n \) solutions of a linear problem any linear combination of these solutions is also a solution i.e. superposition can be applied. In the chapters to follow use will be made of this principle.

A linear equation with a linear boundary condition can be written as:

\[
O(t,x) = 0 \quad \text{and for the boundary condition } \{B(T(x,t))\}_s = b(x_s,t)
\]

where \( O \) and \( B \) are linear operators and \( x_s \) a position on the boundary

For example the operators of the transient heat transfer equation are:

\[
O = \rho c \frac{\partial}{\partial t} - \lambda \frac{\partial^2}{\partial x^2} - \lambda \frac{\partial^2}{\partial y^2} - \lambda \frac{\partial^2}{\partial z^2}
\]

\[
B = \frac{\partial}{\partial n} (n \cdot h_{cv} + h_r) \quad \text{and} \quad b = (n \cdot h_{cv} + h_r)T_{rev}
\]

\( O \) is a linear operator means:

\[
O(sT) = sOT \quad \text{with } s \text{ a constant and}
\]

\[
O(T_1 + T_2) = OT_1 + OT_2
\]

For a linear problem superposition can be applied:

Let \( b = \sum b_n \) and \( OT_n = 0 \) with boundary condition \( (BT_n)_s = b_n \)

then: \( T = \sum T_n \)

2.6.2. The temperature ratio

Risks of condensation and mould growth at the interior surface of a room are determined to a great extend by the relative humidity at the surface. A criterion is that the monthly mean relative humidity must be lower than 80%. This doesn’t mean that then there are no risks any more but that on the basis of the present knowledge the risk is acceptable. Moreover a more pragmatic consideration for this criterion is that with nowadays state of the art of building technology the criterion must be feasible.

To check whether the criterion is not exceeded one needs calculated data about the indoor air temperature, the indoor relative humidity and the lowest surface temperature (at the thermal bridges) during the most critical month. This is too complicated for a standard. A first simplification is to use a standard in- and outdoor climate. If the actual climate is more severe regarding the surface humidity, moisture problems can still arise of course.

The most obvious choice for a standard to prevent surface moisture problems would have been to make a standard for the lowest admissible surface temperature calculated with the standard climate. In the Dutch
building code a dimensionless quantity is chosen instead: the temperature ratio. This quantity is defined as:

$$f_N = \frac{\Theta_{si} - \Theta_e}{\Theta_a - \Theta_e}$$

(2.12)

The advantage of the temperature ratio is that the value is between 0 and 1 and that the factor suggests that if it has a known value the surface temperature can be calculated easily for all values of indoor and outdoor temperature. This is not completely true as we will see.

The calculation of the surface temperature however is not straightforward. The knowledge of indoor and outdoor temperature is often not sufficient but by eq.2.11 it was clear that for 3 bordering temperatures two factors are needed. As a consequence the temperature ratio depends on the chosen temperatures and cannot be used to calculate the surface temperature if more temperatures than $T_a$ and $T_e$ are needed.

In the Dutch code the standard climate is given by the effective outdoor temperature: $\Theta_e = 0^\circ$C and the (reference) indoor air temperature $\Theta_a = 18^\circ$C, chosen in the middle of a room at 1.7 m above the floor. The standard ground temperature is $\Theta_b = 10^\circ$C at 3m below the ground level and also the temperatures in the rooms adjacent to the considered room are standardised for the calculation of $f_N$.

The requirement to the temperature ratio is: $f_N > 0.65$. So $\Theta_{si} > 11.7^\circ$C.

At the reference temperatures and an indoor relative humidity of 53% this means that the R.H. at the surface is: $53 \times 2065/1368 = 80\%$.

Another problem is the indoor surface coefficient of heat transfer. The surface temperature is strongly dependent on these coefficients. Especially in corners, where there is radiation exchange between two or three surfaces and less to the interior of the room, this is a problem. Also behind cupboards the surface temperature is lower. Furthermore, the air temperature near the ceiling is higher than near the floor, the so-called temperature stratification. In order to have a simple standard the last two problems are combined to one: the surface heat transfer coefficients are chosen in such a way that for the calculation always the reference temperature can be used. E.g. a higher air temperature can lead to the same surface temperature as a lower one if a larger surface coefficient is used.

In the code the stratification of the indoor air is taken into account by a smaller surface coefficient near the floor: $h_i = 2 \text{ W/m}^2\text{K}$ than near the ceiling: $h_i = 4 \text{ W/m}^2\text{K}$. It turned out that it was not feasible to require a temperature ratio $>0.65$ for corners with three planes and $h_i = 2 \text{ W/m}^2\text{K}$. So a pragmatic value of $h_i = 4 \text{ W/m}^2\text{K}$ was adopted for this situation. The consequence is that certainly when cupboards are placed in these corners there is still a great mould risk.
2.7. Numerical models of thermal bridges

An analytical solution of the equation enables determination of the temperature at any point in the material. A numerical solution provides the temperature in discrete points: the so-called nodes. The first step at setting up a numerical model is the selection of the nodes. Connecting each node to the nearest ones gives rise to a mesh or grid. The smallest volumes (3D) or areas (2D) are the cells. The interfaces between different materials have to correspond with cell-borders. The accuracy of the simulation depends on the choice of the grid and the number of cells (usually a fine grid is more accurate). At places where large temperature gradients are to be expected (in insulation and near thermal bridges) more temperature nodes are required, so a finer grid has to be chosen.

The next step is deriving a set of algebraic equations for the unknown nodal temperatures \( T_1, T_2, \ldots, T_n \) e.g.:

\[
\sum_{j=1}^{n} a_{ij} T_j = b_i T_{ei} \tag{2.13}
\]

where \( a_{ij} \) and \( b_i \) are constants and \( T_{ei} \) is a known boundary temperature.
This set is solved with the computer. With the calculated temperatures the temperature distribution of the detail can be calculated by interpolation. Also the heat flow density can be calculated (with Fourier’s law) once the temperature distribution is known.

The most applied methods for setting up the algebraic equations are:

- the finite elements method
- the energy balance method

The energy balance method is also called the control volume method and (erroneous) the finite differences method.

The cells with the finite elements method typically have the shape of a tetraeder (3D) or a triangle (2D) that allow very complicated geometries to be modelled.

The cells with energy balance method have rectangular cross-sections (rectangular grid). This is especially convenient for a calculation detail with a rectangular shape. In comparison with the finite elements method the method is much easier to understand physically and easier to program. Each node exchanges heat with 6 adjacent nodes in 3D, with 4 in 2D or with 2 in 1D. This means that in the matrix with elements $a_{ij}$ many elements are zero. For such matrix very quick solution methods are available. If there is radiation exchange between nodes e.g. in an enclosure with air in the structure or in corners this is less simple. Often the problem is simplified to avoid this exchange: e.g. by filling up the enclosure with a fictive material. If there is only conduction

The numerical accuracy can be evaluated by comparison with the analytical solution of the same problem. This can of course only be done if such a solution is obtainable, so for a very simple detail. The conclusion from this comparison might be that no ‘bugs’ are present in the program. The accuracy however if a complicated detail is simulated cannot be assessed by this method.

In order to determine whether the simulation is ‘sufficient’ accurate the result can be compared with the result obtained with a finer grid, e.g. the double number of nodes. For example a criterion can be that the largest difference found by doubling is less than 0.5% of the difference between the highest and lowest nodal temperature.

<table>
<thead>
<tr>
<th>From the energy balance of the shaded area (see figure below) the next equations can be derived:</th>
</tr>
</thead>
</table>
| \[
\frac{1}{2} \left( \Delta y_1 \lambda_{11} + \Delta y_2 \lambda_{21} \right) \frac{T_{i+1,j} - T_{i,j}}{\Delta x_1} + \frac{1}{2} \left( \Delta x_1 \lambda_{11} + \Delta x_2 \lambda_{21} \right) \frac{T_{i+1,j} - T_{i,j}}{\Delta y_1} = \]
| \[
\frac{1}{2} \left( \Delta y_1 \lambda_{21} + \Delta y_2 \lambda_{22} \right) \frac{T_{i+1,j+1} - T_{i+1,j}}{\Delta x_2} + \frac{1}{2} \left( \Delta x_1 \lambda_{12} + \Delta x_2 \lambda_{22} \right) \frac{T_{i+1,j+1} - T_{i,j+1}}{\Delta y_2} = \]
| So the equation for the node $i,j$ is in a concise notation: |
\begin{align*}
G_{i-1,j}T_{i-1,j} + G_{i,j-1}T_{i,j-1} + G_{i+1,j}T_{i+1,j} + G_{i,j+1}T_{i,j+1} - (G_{i-1,j} + G_{i,j-1} + G_{i+1,j} + G_{i,j+1})T_{i,j} &= 0 \\

\text{For a node on the surface the equation is:} \quad &\frac{1}{2} \Delta y \lambda_1 \frac{T_{i+1,j} - T_{i,j}}{\Delta x_1} + \frac{1}{2} (\Delta x_1 \lambda_1 + \Delta x_2 \lambda_2) \frac{T_{i,j} - T_{i-1,j}}{\Delta y} = \\
&\frac{1}{2} \Delta y \lambda_2 \frac{T_{i+1,j} - T_{i,j}}{\Delta x_2} + \frac{1}{2} (\Delta x_1 + \Delta x_2) h_e (T_{i,j} - T_e) \\
&\text{or (j is on the surface):} \\
G_{i,j} T_{i+1,j} + G_{i,j+1} T_{i,j+1} + G_{i+1,j} T_{i+1,j} - (G_{i,j} + G_{i,j+1} + G_{i+1,j} + G_{i,j+1})T_{i,j} = G_{i,j} T_e \\

\text{By interpreting } G_{ij} \text{ as an inverse thermal resistance (a conductance) the equations are exactly the same as the ones that can be derived with a resistance circuit.}
\end{align*}
Questions

What are isotherms? What are heat flow lines? What can be concluded from heat flow lines? Sketch these lines for a corner between two massive exterior walls. Do this for an external and an internal corner. Draw the isotherms and heat flow lines for a balcony floor that is part of the interior floor.

What problem do you expect with walls that contain thermal bridges? How can thermal bridges be prevented? Mention measures to reduce the problems (or the main problem) afterwards.

Give the definition of the temperature ratio.

Why was the temperature ratio introduced?

The temperature ratio depends on the building detail but also on other factors. Mention these factors and indicate what influence they have on the temperature ratio at a thermal bridge shaped by a 3D corner.

Why do the surface heat transfer coefficients used for the calculation of the temperature ratio factor have such ‘strange’ values?

With the simplified method the equivalent thermal transmittance of a wall with a thermal bridge can be estimated. Which physical simplifications are made for the calculation of the minimum and the maximum value and why is that value minimum or maximum?

What are the conditions for assuming that a thermal bridge is isotherm? (material properties etc?)

Give examples of thermal bridges that are isothermal.

How can you see from the isotherms where the insulation material is situated?

How can you see from the heat flow lines where the thermal bridge is situated?

Give at least 3 problems in the field of building physics that you come across with interior insulation.
3. One-dimensional transient heat conduction

3.1. Introduction

Boundary conditions change all the time. Because heat is stored when the temperature rises and is released when the temperature decreases the temperature distribution and the heat flow in a structure depend not only on the instantaneous values but also of previous values. The influence of these values decreases as they are more back in time. Also near the boundary the variations are larger than deeper in the material.

Many problems in building physics can successfully be treated one-dimensional. An example is the calculation of the heating or cooling demand. Another application is the calculation of the thermal expansion resulting from temperature variations.

The linear one-dimensional equation for transient heat conduction is:

\[
\frac{∂T}{∂t} = a \frac{∂^2T}{∂x^2}
\]

where\(a = \frac{λ}{ρc}\) the thermal diffusivity \([m^2/s]\)

In this chapter we treat some analytical solutions with special initial and boundary conditions and finally some numerical methods.

Analytical solutions give a clear insight in the influence of material properties, dimensions, and boundary conditions. Sometimes they are good approximations of the exact solution. Additionally they can be used to check computer models (analytical validation). So notwithstanding the fact that nowadays the computer is dominant and analytical solutions are seldom used in practice, they are treated here.

We confine the treatment of analytical solutions to two types of boundary conditions: step changes in time and the sinusoidally varying conditions. Examples of step changes are: grasping a cold object (contact temperature), sunshine suddenly falling on a facade, changing the temperature (by the thermostat) in a room. A sinusoidal (cyclic) variation is an approximation of a periodic variation: the day-night cycle of temperature and solar radiation, the annual cycle.
Important is:

- Insight in transient heat transfer and the influence of material properties.
- Knowledge of the background of computer models and their possible errors or accuracy.
- Capable of deriving simple thermal models

\[
q = \frac{T_1 - T_2}{R} \\
R = \frac{d}{\lambda}
\]

\[
q_1 - q_2 = C \frac{dT}{dt}
\]

\[
C = \rho c d
\]

electrical resistance (R) ↔ thermal resistance:

\[E = i \times R\] is analogous to: \[\Delta T = q/h\]

electrical capacitance (condensator) ↔ surface heat capacitance

\[i = C \frac{dE}{dt}\] is analogous to: \[q = \rho c d \frac{dT}{dt}\]
3.2. Step changes

3.2.1. A volume of material with a very high heat conductivity

In Chapter 2 approximate solutions for a thermal bridge with a very high thermal conductivity were derived. Also for the transient heat transfer this can be done.

Suppose that the effective temperature near a volume of high conductive material and an uniform temperature $T_0$ changes at time $t=0$ with a step function from $T_0$ to $T_{rev}$, i.e. that the initial and boundary conditions are:

- $t < 0$ \[ T = T_0 \]
- $t \geq 0$ \[ -\lambda \frac{dT}{dn} = h(T_{rev} - T) \]

Here $n$: normal on the surface pointed into the material.

If the temperature in the material is also uniform for $t>0$ (consequence of the high thermal conductivity), then the surface temperature equals the temperature in the material. So the equation for the transient heat transfer becomes ($A=$surface area $A$ and $V=$volume):

\[ \frac{Ah}{V} (T_{rev} - T) = \rho c V \frac{dT}{dt} \] \hfill (3.2)

So for a slab (thickness $d$, volume $V = A.d$) perfectly insulated at one side the equation is:

\[ h (T_{rev} - T) = C \frac{dT}{dt} \] \hfill (3.3)

where: $C = \rho c d$ the surface heat capacitance

The solution of this first order differential equation for a stepwise change ($T_{rev} - T_0$) is:

\[ T = T_{rev} + (T_0 - T_{rev}) \exp \left( -\frac{h}{C} t \right) \] \hfill (3.4)

$C/h$ is the time constant. After one time-constant the temperature difference between the material and the environment changes to $1/e = 0.37$ of the value at the beginning. So a small time constant (large $h$, small $C$) stands for a quick temperature change.

The thermal circuit of the equation above is a resistance with a capacitance in series. This representation of equations by means of thermal circuits is very convenient and we will often use this instead of writing down the differential equations.
If we integrate the general equation over the volume of the material then after changing the order of integration and differentiation we get:

\[
\lambda \int \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \, dx \, dy \, dz = \rho c \frac{d}{dt} \int T \, dx \, dy \, dz
\]

or:

\[
- \lambda \int \frac{\partial T}{\partial n} \, dA = \rho c V \frac{dT}{dt} = Ah(T_{rev} - T)
\]

In the l.h.s. the heat flow densities to the surface are integrated: so this is the total heat flow to the material. The r.h.s. of the equation represents the change of stored heat. The equation is just the heat balance.

The validity of the assumption that the temperature differences in the material can be neglected compared to the difference between surface temperature and the environment depends on the ratio of the thermal resistance in the material \((d/\lambda)\) with a characteristic length \(d\) and the surface resistance \(1/h\).

The condition is \(hd/\lambda<<1\) or \(Bi<<1\) (Biotnumber)

So a small size or a large thermal conductivity is required.

3.2.2. A temperature step at the surface of a semi-infinite thick slab

The initial and boundary conditions are:

\[
\begin{align*}
    t < 0 & \quad T(x, t) = T_0 \\
    x = 0 \text{ and } t \geq 0 & \quad T(0, t) = T_o + \Delta T_s \\
    x \rightarrow \infty \text{ and } t \geq 0 & \quad T(\infty, t) = T_0
\end{align*}
\]

The solution is:

\[
T(x, t) = T_0 + \Delta T_s \operatorname{erfc} \left( \frac{x}{2\sqrt{at}} \right)
\]

(3.5)

where \(\operatorname{erfc}(z)\) is the complementary error integral of Gauss.

The complementary error integral of Gauss is defined as:

\[
\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} \exp \left( -\xi^2 \right) \, d\xi
\]

So \(\operatorname{erfc}(0) = 1\) and \(\operatorname{erfc}(\infty) = 0\)

N.B.: A good approximation is (error < 1%):

\[
\operatorname{erfc}(z) = \frac{\sqrt{\pi}}{2z + \sqrt{\pi} + \sqrt{\pi} z^2 + \pi} \exp \left( -z^2 \right)
\]
At a distance d from the surface the time needed for a temperature rise of 10% of the step on the surface is found from:

\[
\text{erfc} \left( \frac{d}{2\sqrt{at}} \right) = 0.1 \quad \text{or} \quad t_f = \frac{0.18d^2}{a}
\]

Fourier number

Reversibly the validity of the solution (3.5) for a slab with finite thickness d is restricted to a limited time \( t < t_f \) counted from the start of the step. This condition can be expressed in a criterion for the so-called Fourier number \( Fo \) of a slab that can be considered infinite thick:

\[
Fo = \frac{at_f}{d^2} \quad \text{So Fo}<0.18 \quad (3.6)
\]

At the surface \((x = 0)\) the density of heat flow rate is:

\[
q_s = -\lambda \left( \frac{\partial T}{\partial x} \right)_{x=0} = \frac{b \Delta T_s}{\sqrt{\pi t}}
\]

(3.7)

where \( b \) = thermal effusivity: \( b = \sqrt{\lambda \rho c} \quad [\text{Ws}^{1/2}/\text{m}^2\text{K}] \)

At time \( t = 0 \) the density of heat flow rate at the surface is infinite in order to increase the surface temperature in infinitesimal time. In the real world however, this increase will take always some time (why?).

The amount of stored heat is:

\[
Q(t) = \frac{A}{\sqrt{\pi}} \left( b \Delta T_s \right) \int_0^t q_s \, dt = \frac{A}{\sqrt{\pi}} \left( b \Delta T_s \right) \sqrt{t}
\]

(3.8)

Where \( A \) = the surface area.
Conclusions:

- The solution at \( x_1, t_1 \) is equal to that at \( x_2, t_2 \) if \( t_2 = (x_2^2/x_1^2) \ t_1 \), e.g. if \( x_2 = 2x_1 \) and \( t_2 = 4t_1 \) the same temperature is found. In order to get the same temperature deeper into the material one has to wait much longer.
- The thermal diffusivity determines the relative distribution of the temperature. The larger ‘\( a \)’ the quicker heat is transferred to the backside. For example the fire retardation of wood (\( a=0.14 \times 10^{-6} \text{ m}^2/\text{s} \)) is much larger than the one of aluminium (\( a = 80 \times 10^{-6} \text{ m}^2/\text{s} \)). In practice wood is also thicker what enhances this effect.
- In the expressions for \( q_s \) and \( Q(t) \) the thermal effusivity (\( b \)) appears. So at the surface not a (thermal diffusivity) but the thermal effusivity is the important material property.
- The amount of stored heat \( Q(t) \) increases proportional to \( \sqrt{t} \).

If two materials with initial uniform temperatures \( T_1 \) and \( T_2 \) are placed in contact at their free surface, heat will be transferred from the material with the highest temperature (e.g. material 1) to the other material. So the temperature at the interface changes step-wise. With eq.3.7 the interfacial \( (T_c) \) can be derived by equating the heat flow of material 1 with the one of material 2:

\[
\frac{b_1 \Delta T_1}{\sqrt{\pi t}} = \frac{b_2 \Delta T_2}{\sqrt{\pi t}} \quad \text{so} \quad b_1(T_1 - T_c) = b_2(T_c - T_2)
\]

\[
T_c = \frac{b_1 T_1 + b_2 T_2}{b_1 + b_2} \quad \text{(3.9)}
\]

So the interfacial temperature is independent of time, at least as the assumption semi-infinite is valid. So the criterion is with eq.3.6:

\[
t < \min \left( \frac{0.18d_1^2}{a_1}, \frac{0.18d_2^2}{a_2} \right)
\]

The thermal effusivity turns out to be responsible for the temperature one feels at touching a material. Polystyrene-foam feels warm (small \( b \), so the hand-temperature is felt) and metal feels cold (large \( b \), so the temperature of the metal is felt).

The comfort sensation one feels at walking on bare feet (and shoed feet) depends not only on the interfacial temperature but also on the heat flow leaving the feet. The quality classification of floors depends on the density of heat flow rate after 60 sec \( (Q_1) \) or after 600 sec \( (Q_{10}) \). In the form of a formula \( Q_1 \) is:

\[
Q_1 = \frac{2 b_1 \sqrt{60}}{\sqrt{\pi}} (T_1 - T_c) = \frac{2 b_1 b_2 (T_1 - T_2) \sqrt{60}}{(b_1 + b_2) \sqrt{\pi}}
\]
Example: foot temperature $\Theta_1 = 31 ^\circ C$, thermal effusivity of foot $b_1 = 1200 \, J/m^2Ks^{1/2}$.

Class ‘good’ $Q_1<38 \times 10^3 \, J/m^2$. Comfortable is $\Theta_2 = 18 ^\circ C$ so $b_2 <450 \, J/m^2Ks^{1/2}$. (see figure)

Because after 600 sec the assumption semi-infinite for the floor covering (carpet, parquet) is not valid anymore the structure below plays a part. This is the reason why for a certain $Q_{10}$ the $Q_1$ of the floorcovering can be smaller than expected by the given theory and is in such case the quality class based on $Q_{10}$.

### Q-values of floor coverings (at $T_1 = 18 ^\circ C$)

<table>
<thead>
<tr>
<th>floor-covering</th>
<th>$Q_1 (kJ/m^2)$</th>
<th>$Q_{10} (kJ/m^2)$</th>
<th>class</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-6 mm cork linoleum</td>
<td>31.5</td>
<td>168-189</td>
<td>1</td>
</tr>
<tr>
<td>2.5 mm lino on heavy floor</td>
<td>49</td>
<td>348</td>
<td>3</td>
</tr>
<tr>
<td>2.5 mm lino on light floor</td>
<td>56</td>
<td>369</td>
<td>3</td>
</tr>
<tr>
<td>8 mm velour (wool+perlon)</td>
<td>17</td>
<td>97</td>
<td>1</td>
</tr>
<tr>
<td>10 mm coconut fibre</td>
<td>15.5</td>
<td>71</td>
<td>1</td>
</tr>
<tr>
<td>pinewood parquet</td>
<td>29.5</td>
<td>124</td>
<td>1</td>
</tr>
<tr>
<td>8 mm parquet on concrete</td>
<td>35.5</td>
<td>162</td>
<td>1</td>
</tr>
<tr>
<td>Beech wood parquet</td>
<td>36.5</td>
<td>165</td>
<td>1</td>
</tr>
<tr>
<td>Oak parquet</td>
<td>36</td>
<td>183</td>
<td>1</td>
</tr>
<tr>
<td>Lightweight plaster floor 1200 kg/m$^3$</td>
<td>36</td>
<td>246</td>
<td>2</td>
</tr>
<tr>
<td>Anhydrite plaster floor 1000 kg/m$^3$</td>
<td>48.5</td>
<td>210</td>
<td>2</td>
</tr>
<tr>
<td>3 layers of woodwool cement plates</td>
<td>52.5</td>
<td>348</td>
<td>3</td>
</tr>
<tr>
<td>Asphalt plates</td>
<td>63.5</td>
<td>437</td>
<td>4</td>
</tr>
<tr>
<td>Reinforced concrete</td>
<td>74</td>
<td>500</td>
<td>4</td>
</tr>
</tbody>
</table>
Concrete $a = 10^{-6} \text{m}^2/\text{s}$, $b = 2000 \text{J/m}^2\text{Ks}^{1/2}$, $\Theta = 10^\circ \text{C}$, $\Theta_c = 17.9^\circ \text{C}$, $Q_1 = 138 \text{kJ/m}^2$, $Q_{10} = 435 \text{kJ/m}^2$.

Insulation foam $a = 10^{-6} \text{m}^2/\text{s}$, $b = 40 \text{J/m}^2\text{Ks}^{1/2}$, $\Theta = 10^\circ \text{C}$, $\Theta_c = 30.3^\circ \text{C}$, $Q_1 = 7 \text{kJ/m}^2$, $Q_{10} = 22 \text{kJ/m}^2$.

The foam feels much warmer than the concrete, but more important is that the heat flow from the foot is smaller by a factor 20. At 18 mm depth and 60 sec after touching, the temperature of both materials has changed 10% of the temperature step at the interface.

### 3.2.3. A step in the density of heat flow rate at the surface of a semi-infinite thick slab

The initial and boundary conditions are:

- $t < 0$ \quad $T(x, t) = T_0$,
- $x = 0$ en $t \geq 0$ \quad $-\lambda \frac{\partial T}{\partial x} = q_s$,
- $x \to \infty$ and $t \geq 0$ \quad $T(\infty, t) = T_0$.

Solution:

$$T = T_0 + q_s \frac{2\sqrt{t}}{b} \left[ \frac{1}{\sqrt{\pi}} \exp \left( -\frac{x^2}{4at} \right) - \frac{x}{2\sqrt{at}} \text{erfc} \left( \frac{x}{2\sqrt{at}} \right) \right] \quad (3.10)$$

Hence for $x = 0$ \quad $T_s - T_0 = \frac{2\sqrt{t}}{\sqrt{\pi} b} q_s \quad (3.11)$

Assuming that the wall can be considered ‘infinite’ thick when the temperature increase at the other side is less than 10% of the surface temperature, the criterion for the Fourier number ($Fo$) is:

$$t_f = \frac{0.27d^2}{a} \quad \text{and} \quad Fo < 0.27$$

Just as in section 3.2.2 the relation between surface temperature and the density of heat flow rate depends on the thermal effusivity and the distribution of the temperature in the material on the thermal diffusivity.

The solution of the problem above in cylinder coordinates can be used for the experimental determination of the thermal conductivity ($\lambda$) and the specific heat capacity by volume ($pc$) of a material. A very thin tightly stretched wire is put in a material and at time $t = 0$ an electric current is applied. If the dissipated power is held constant a stepwise change of the heat flow density is attained. By measuring at known distances from the wire the temperature in the material $a$ and $b$ can be deduced and hence the $\lambda$ and $pc$. The advantage of this (transient) method is the short time needed for a measurement. This is required for measurements in moist materials, viz. when moist materials are heated a
redistribution of moisture takes place and water can also evaporate so the heat conductivity is changing during the measurement. A disadvantage is the sensitivity of the result on the position of the wire and the thermopiles in the material so that the accuracy is often poor.

### 3.2.4. A step in the effective temperature at the surface of an semi-infinite thick slab

A step in the effective temperature can be caused by sudden solar radiation on a wall. Initially the heat flow to the wall equals the product of the surface coefficient and the difference between effective temperature and the initial surface temperature i.e. the solution resembles the one for a heat flow step (eq.3.10). Then the surface temperature will rise and the heat flow will decrease. After some time the heat flow is so small that there is hardly any difference between the surface temperature and the effective temperature. In that case the solution is converges to the solution with a surface temperature step (eq.3.5).

The initial and boundary conditions for this problem are:

\[
\begin{align*}
    t < 0 & \quad T(x, t) = T_0 \\
    x = 0 \text{ en } t \geq 0 & \quad -\lambda \frac{\partial T}{\partial x} = h (T_{rev} - T) \\
    t \to \infty \text{ en } t \geq 0 & \quad T(\infty, t) = T_0
\end{align*}
\]

The solution of this problem is:

\[
T = T_0 + \left[T_{rev} - T_0\right] \left[\text{erfc} \left(\frac{x}{\sqrt{2at}}\right) - \exp \left(\frac{hx}{\lambda} + \frac{h^2t}{b^2}\right) \text{erfc} \left(\frac{x}{2\sqrt{at}} + \frac{h\sqrt{t}}{b}\right)\right]
\]

The wall can be considered infinite thick if: \( t < \frac{0.27 d^2}{a} \) or \( \text{Fo} < 0.27 \)

The equation for the surface temperature is:

\[
T_s = T_0 + (T_{rev} - T_0) \left[1 - \exp \left(\frac{h^2t}{b^2}\right) \text{erfc} \left(\frac{h\sqrt{t}}{b}\right)\right]
\]

and for the density of heat flow rate:

\[
q_s = h(T_{rev} - T_0) \exp \left(\frac{h^2t}{b^2}\right) \text{erfc} \left(\frac{h\sqrt{t}}{b}\right)
\]

Hence the ratio of the density of heat flow rate and the initial \((t=0)\) density of heat flow rate is solely a function of \(h\sqrt{t}/b\).

With these equations an estimate can be made for the surface temperature of a wall exposed to a step in the effective temperature at the other side. With fire the surface temperature will increase more
rapidly the more the thermal effusivity is smaller and the surface coefficient is larger. The temperature rises at the backside quicker as the thermal diffusivity is larger and the thickness smaller.

The stored heat is:

\[ Q(t) = \int_0^1 q_s \, dt = \left( T_{\text{rev}} - T_0 \right) \frac{b^2}{h} \left[ \exp \left( \frac{h^2 t}{b^2} \right) \text{erfc} \left( \frac{h \sqrt{t}}{b} \right) - 1 + \frac{2}{\sqrt{\pi}} \frac{h \sqrt{t}}{b} \right] \quad (3.15) \]

At the surface a concrete wall (d = 0.1 m, \( \lambda = 2 \) W/mK, c = 840 J/kgK, \( \rho = 2500 \) kg/m³) the effective temperature increases from 0 to 20°C. The backside is insulated very well (R = \( \infty \)), h = 8 W/m²K.

Calculate the increase of the surface temperature after 1000, 10,000 and 50,000 sec.

Criterion: \( t < \frac{0.27 \, d^2}{a} = \frac{0.27 \cdot 10^{-2}}{0.95 \cdot 10^{-6}} = 2800 \) sec

\( t = 1000 \) sec eq.3.13: \( \frac{T_s - T_0}{T_{\text{rev}} - T_0} = 0.12 \) exact 0.13

\( t = 10,000 \) sec eq.3.13: \( \frac{T_s - T_0}{T_{\text{rev}} - T_0} = 0.32 \) exact 0.4

\( t = 50,000 \) sec eq.3.13: \( \frac{T_s - T_0}{T_{\text{rev}} - T_0} = 0.53 \) exact 0.75

The last two don’t satisfy the criterion for infinite thick with as a consequence a clear difference with the exact solution.

For \( t \to 0 \) the solution for the constant density of heat flow rate holds and for \( t \to \infty \) the solution for the constant surface temperature. (eq.3.7 and 3.11)

So:

\[ \frac{\sqrt{\pi t}}{b} > \frac{T_s - T_0}{q_s} > \frac{2\sqrt{t}}{\sqrt{\pi b}} \]

This is satisfied by the general solution. The proof is simple with the approximation of \( \text{erfc}(z) \exp(z^2) \) (see section 3.2.1)
3.3. Periodic variations

3.3.1. Fourier analysis and complex numbers

Until now only step changes are treated i.e. a discontinuous change at a certain time. In building physics another very important class of changes are the periodic ones: after a certain time (the period) the quantity has the same value again. The main periods for repetition are the annual ones with a period of 365 days and the daily ones with a period of 24 hours. In reality the changes are not exactly periodic (every year is different, every day is different) but only as an approximation. In a mathematical representation a periodic change of temperature means:

\[ T(t + nt_0) = T(t) \]  \hspace{1cm} (3.16)

where:  
- \( n \) = an arbitrary natural number  
- \( t_0 \) = period

A periodic function can be written as the sum of sinusoidal (or cyclic) functions with different frequencies that are always a multiple of the base frequency (similar to a noise spectrum): a so-called Fourier series.

Fourier series can be derived for the outdoor temperature, the global radiation, the vapour pressure etc. The amplitude spectrum shows peaks at the daily cycle (\( t_0/n = 24 \times 3600 \) sec) and at the annual cycle (\( t_0 = 365 \times 24 \times 3600 \)). If all small components of the spectrum are neglected an approximation for the transient course of the climatological quantities is found. In the representation of a formula:

\[ Y = Y_{\text{mean}} - \frac{a - \cos(2\pi \frac{t-t_0}{8760}) - (c - \cos(2\pi \frac{t-t_1}{8760}))\cos(2\pi \frac{t-t_2}{24})}{2} \]  \hspace{1cm} (3.17)

The coefficients for De Bilt (NL) are listed in the table below:

<table>
<thead>
<tr>
<th>( Y )</th>
<th>( Y_{\text{mean}} )</th>
<th>( a )</th>
<th>( t_0 )</th>
<th>( B )</th>
<th>( t_1 )</th>
<th>( C )</th>
<th>( T_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_e )</td>
<td>9.41</td>
<td>7.35</td>
<td>533</td>
<td>1.5</td>
<td>41</td>
<td>2.6</td>
<td>1.75</td>
</tr>
<tr>
<td>( E_e )</td>
<td>111.5</td>
<td>97.5</td>
<td>-222</td>
<td>127.4</td>
<td>219</td>
<td>166</td>
<td>23.66</td>
</tr>
<tr>
<td>( p_e )</td>
<td>1008</td>
<td>407</td>
<td>763</td>
<td>6</td>
<td>-2066</td>
<td>31</td>
<td>1.78</td>
</tr>
</tbody>
</table>

With these coefficients and eq.3.17 the figures below can be made:
In the adjacent figures the first three circular functions of a blockshaped periodic function are depicted.

A periodic function, for which yields \( \int_{0}^{t_0} |f(t)| \, dt < \infty \), can be written as the sum of sine and cosine functions, the Fourier series:

\[
T(t) = T + \sum_{n=1}^{\infty} \left( a_n \cos \omega_n t + b_n \sin \omega_n t \right)
\]

\( \omega_n / 2\pi = n / t_0 \) is the frequency (a multiple of the base frequency \( 1 / t_0 \)). The coefficients \( a_n \) and \( b_n \) are called the Fourier coefficients. They result from the Euler equations:

\[
a_n = \frac{2}{t_0} \int_{0}^{t_0} T(t) \cos (\omega_n t) \, dt \quad ; \quad b_n = \frac{2}{t_0} \int_{0}^{t_0} T(t) \sin (\omega_n t) \, dt
\]

\[
\bar{T} = \frac{1}{t_0} \int_{0}^{t_0} T(t) \, dt
\]

Instead of writing the Fourier series with sine and cosine functions also solely cosine functions can be used:

\[
T(t) = \bar{T} + \sum_{n=1}^{\infty} \hat{T}_n \cos (\omega_n t - \phi_n)
\]

\( \hat{T}_n(\omega) \) is the amplitude spectrum and \( \phi_n(\omega) \) is the phase spectrum.

Proof the following equation:

\[
\int_{0}^{t_0} \cos \left( \frac{2\pi m}{t_0} t \right) \cos \left( \frac{2\pi n}{t_0} t \right) \, dt = \begin{cases} 
0 & \text{if } m \neq n \\
\frac{t_0}{2} & \text{if } m = n
\end{cases}
\]

Complex number: \( z = a + bj \)

where: \( j^2 = -1 \)

\( a = \text{Re}(z) \) (real part)

\( b = \text{Im}(z) \) (imaginary part)

Real numbers can be represented by a point on a line. Complex numbers have a representation of a point on a plane: the complex plane. The y-value of this point is the imaginary part and the x-value the real part. So addition of complex numbers is the addition of two vectors. The y-axis is called the imaginary axis and the x-axis the real axis.

Multiplication of complex numbers:

\( (a + bj)(c + dj) = ac - bd + j(bc + ad) \)

Division of complex numbers:

\[
\frac{a + bj}{c + dj} = \frac{(a + bj)(c - dj)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + j \frac{bc - ad}{c^2 + d^2}
\]

Show that: \( \sqrt{j} = (1+j) \frac{1}{2} \sqrt{2} \)
The Euler formula is:
\[ z = |z| e^{i \phi} = a + bj \]

with: \[ |z| = \sqrt{a^2 + b^2} \]
and \[ \tan \phi = \frac{b}{a} \]

The modulus (==absolute value) \( |z| \) is the length of the vector in the complex plane and \( \phi \) is the angle of the vector with the positive real axis.

Suppose that the temperature can be written as:
\[ T(t) = \hat{T} + \tilde{T}(t) = \tilde{T} + \hat{T} \cos (\omega t - \phi) \]

In the complex notation this is:
\[ T(t) = \hat{T} + \tilde{T}(t) = \tilde{T} + \hat{T} \Re \{ \cos (\omega t - \phi) + j \sin (\omega t - \phi) \} \]

So: \( \tilde{T} = \Re \{ \hat{T} e^{j(\omega t - \phi)} \} \)

In the complex plane this function for successive times turns out to be the real part of a point that circles around the origin. The radius of the circle is the amplitude of the temperature, the angular velocity equals \( \omega \) and the angle with the real axis at \( t = 0 \) equals \( \phi \).

With the complex notation the equations for the Fourier series and coefficients can be written more compact:
\[ T(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n t} \quad \text{with} \quad c_n = \frac{1}{T_0} \int_{0}^{T_0} T(t) e^{-j\omega_n t} dt \]

Show that:
\[ \Re(c_n) = \Re(c_{-n}) \text{ and } \Im(c_n) = \Im(c_{-n}) \]
And: \( a_n = 2\Re(c_n) \) and \( b_n = -2\Im(c_n) \) for \( n > 0 \)

### 3.3.2. A volume with a high heat conductivity

In section 3.2.1 is treated that the thermal circuit of this problem consists of a heat capacitance \( C \) and a thermal resistance \( R \) in series.

Let the heat flow be:
\[ \tilde{q} = \dot{q} \cos \omega t \quad (3.18) \]

Then the temperature difference across the thermal resistance is:
\[ \Delta \tilde{T} = \tilde{q} \cdot R = \dot{q} R \cos \omega t \]
For the capacitance yields: \[ \ddot{q} = C \frac{dT}{dt} \] (3.19)

So with eq.3.18 the wall temperature is:

\[ \bar{T}_s = \frac{\dot{q}}{\omega C} \sin \omega t \]

Comparing this result with the result of the temperature drop of a resistance leads to the conclusion that a capacitance has an apparent resistance of \(1/\omega C\). Hence this resistance depends on the frequency. Moreover it is not a real resistance as the phase of the temperature is \(90^\circ\) behind the phase of the heat flow (in time \(t_0/4\)). As a consequence of this phase-shift the apparent resistance of the capacitance and the real resistance cannot be added. In this example the total temperature is:

\[ \bar{T}_t = \bar{T}_s + \Delta \bar{T} = \dot{q} \left( \frac{1}{\omega C} \sin \omega t + R \cos \omega t \right) \]

\[ \bar{T}_t = \sqrt{\bar{R}^2 + \left( \frac{1}{\omega C} \right)^2} \dot{q} \cos(\omega t + \phi) \]

\[ \text{met} \quad \tan \phi = -\frac{1}{\omega RC} \] (3.20)

The phase shift between total temperature and heat flow density is between 0 and \(90^\circ\) and the apparent resistance is: \(\sqrt{\bar{R}^2 + 1/\omega^2 C^2}\).

The same result can be obtained much easier with complex numbers:

\[ \bar{q} = \dot{q} e^{j\omega t} \quad \text{and} \quad \bar{T} = \frac{\dot{q}}{j\omega C} \]

The impedance (=apparent resistance) of the capacitance \(C\) has an imaginary value (see below):

\[ X = \frac{\bar{T}}{\bar{q}} = \frac{1}{j\omega C} \]

The total temperature is:

\[ \bar{T}_t = \left( R + \frac{1}{j\omega C} \right) \dot{q} e^{j\omega t} \]

The real part of this solution yields exactly formula (3.20). By using complex functions the calculation with sine and cosine functions can be avoided. This simplifies the calculation.
A heat capacitance is implicitly defined by: \( \ddot{q} = C \frac{dT}{dt} \).

Suppose \( \ddot{T} = \hat{T} \cos \omega t \)

So the density of heat flow rate is:

\( \ddot{q} = -\omega \dot{C} \hat{T} \sin \omega t \)

Complex: \( \ddot{T} = \hat{T} e^{j\omega t} \) and \( \ddot{q} = j\omega \dot{C} \hat{T} e^{j\omega t} \)

The real part of \( q \) is the correct solution.

The quotient of the temperature and the heat flow rate results is a kind of thermal resistance:

\[
X = \frac{\ddot{T}}{\ddot{q}} = \frac{1}{j\omega \dot{C}}
\]

This complex resistance is called impedance. Note that if the same quotient was calculated with the real temperature and heat flow a time dependent function would have been found! With the arithmetic of complex numbers a circuit of heat capacitances and thermal resistances can easily be solved.

### 3.3.3. Cyclic surface temperature variations of a semi-infinite thick slab

Suppose that the initial and boundary conditions are:

\[
x = 0 \quad T(0, t) = \hat{T}_s e^{j\omega t} + T_0
\]

\[
x \rightarrow \infty \quad T(\infty, t) = T_0
\]

Solution:

\[
T(x, t) = \hat{T} e^{j\omega t} - (1 + j)x/d^* + T_0
\]

Where \( d^* = \sqrt{\frac{at_0}{\pi}} \) \( (3.21) \)

The real part is:

\[
T(x, t) = \hat{T}_s e^{x/d^*} \cos(\omega t - x/d^*) + T_0 \quad (3.22)
\]

This solution has the appearance of an attenuated vibration: the further away from the surface the smaller the amplitude of the variation. The amplitude for \( x = d^* \) is only \( 1/e \) of the value on the surface. We shall call \( d^* \) the penetration depth. At \( x = 3 \, d^* \) the amplitude is only 5\% of the value on the surface. Because \( d^* \) gets smaller for a higher frequency \( (t_0 \) is smaller) slow variations penetrate deeper than quick ones. The penetration depth of the annual cycle is a factor \( \sqrt{365} = 19 \) times longer than the daily cycle.
E.g. at a depth of 4 to 14 m in the ground only 5% of the seasonal variation is left while the depth for the same attenuation of the daily variation is 20 to 70 cm. For these depths the annual mean temperature is usually supposed to be constant (\( T_0 \)). In The Netherlands this is \( \Theta_0 = 10^\circ C \). So this is also the annual mean surface temperature. Not only the attenuation but also the phase shift increases the further away from the surface. The maximum at the surface (e.g. at \( t = 0 \)) arrives later at greater depths (\( t > 0 \)). The wave has a certain time lag. This time lag is given by eq.3.22:

\[
\cos (\omega t - x/d^*) = \cos \omega (t - \Delta t) \tag{3.23}
\]

So:

\[
\Delta t = \frac{x}{2\pi d} t_0 \tag{3.24}
\]

The propagation velocity of the wave is:

\[
v = \frac{x}{\Delta t} = \frac{2\pi d^*}{t_0} \tag{3.25}
\]

Suppose one wants to chose the thickness of a homogeneous wall in such a way that the maximum surface temperature at the interior appears 12 hours later than the maximum at the exterior surface. How thick should the wall be?

\[
\Delta t = \frac{x}{2\pi d^*} t_0 = \frac{1}{2} t_0 \quad \text{hence} \quad x = \pi d^*
\]

If for \( t_0 = 24 \times 3600 \text{ sec} \) the value of \( d^* \) is 13 cm (e.g. adobe) then the thickness must be approximately 40 cm. With this thickness the amplitude of the variation at the inside is very small (< 5%). So instead of delaying, the variation is practically levelled out!
The varying heat flow to the surface of a semi-infinite thick wall is:

\[ \tilde{q}_s = -\lambda \left( \frac{dT}{dx} \right)_{x=0} = (1 + j) \frac{\lambda}{d} \bar{T} e^{j\omega t} = (1 + j) b \sqrt{\frac{\pi}{t_0}} \]  

(3.26)

Also here the thermal effusivity turns out to be the material property that determines the ratio heat flow and temperature (compare 3.2.1 and 3.2.2).

The impedance of this wall is: (eq.3.21 and eq.3.26).

\[ Z = \frac{T_s}{\tilde{q}_s} = \frac{d^*}{\lambda (1 + j)} = \frac{d^* (1 - j)}{2\lambda} = \frac{d^*}{2\lambda} + \frac{d^*}{j2\lambda} \]

Comparing this result with the result of the preceding section then one can conclude that for the calculation of the surface temperature and the heat flow at the surface a semi-infinite slab can be represented by a resistance and capacitance in series.

\[ R = \frac{d^*}{2\lambda} = \frac{1}{2} \frac{t_0}{b \sqrt{\pi}} \]  

(3.27)

\[ \frac{1}{j\omega C} = \frac{d^*}{j2\lambda} \]  

\[ C = \frac{2\lambda}{\omega d^*} = \frac{\rho c d^*}{b} \frac{t_0}{\pi} \]  

(3.28)

For the capacitance the apparent thickness of the wall is \( d^* \) and for the resistance this is \( d^*/2 \).

Eq.3.26 also shows that the phase shift between the heat flow density and the temperature at the surface equals 45°, so 1/8 period. Usually this is even less.

Summarizing the results for a wall with a finite thickness:
- the variations of the heat flow through an exterior wall to the interior by exterior temperature variations are delayed. The time lag can be, dependant on \( \frac{d_0}{d^*} \), very long e.g. 12 hours
- the time lag at the same side between interior temperature and surface temperature is at maximum \( t_0/8 \) hours. Heat caused by solar radiation absorbed at the floor can only be stored in the walls for a limited time: less than 24/8=3 hours
- the heat storage in a room hardly increases by constructing the walls at the inside of the insulation thicker than \( d^* \).

The boundary conditions suggest a solution as follows:

\[ T(x, t) = \Theta(x) e^{j\omega t} + T_0 \]

where \( \Theta(x) = an as yet unknown complex function. \)

Substitution in the linear one-dimensional transient heat conduction equation (section 3.1) yields:
\[ \rho c j \omega \Theta(x)e^{j\omega t} = \lambda \Theta e^{j\omega t} \frac{d^2}{dx^2} \Theta(x) \]

or
\[ \frac{j \rho c \omega}{\lambda} \Theta(x) = \frac{d^2}{dx^2} \Theta(x) \]

This proves that the proposed solution suffices (only one independent variable: x). The solution of this 2nd order differential equation can be written as:
\[ \Theta(x) = \Theta(0)e^{kx} \]

After substitution we find:
\[ \frac{j \rho c \omega}{\lambda} = k^2 \quad \Rightarrow \quad k = \pm \sqrt{\frac{j \rho c \omega}{\lambda}} \]

Because \((1 + j)^2 = 1 + 2j - 1 = 2j\):
\[ k = \pm (1 + j) \sqrt{\frac{\rho c \omega}{2\lambda}} = \pm (1 + j) \sqrt{\frac{\pi}{at_0}} \quad \text{with} \quad a = \frac{\lambda}{\rho c} \]

The property k has the dimension of \(m^{-1}\), \((at_0/\pi)^{1/2}\) has the dimension of a thickness. We denote this thickness with \(d^*\). The solution then reads:
\[ T(x,t) = e^{j\omega t} \left[ \Theta_1(0)e^{(1+j)x/d^*} + \Theta_2(0)e^{-(1+j)x/d^*} \right] + T_0 \]

From \(T(x,t) = T_0\) follows \(\Theta_1(0) = 0\).

With the boundary conditions \(\Theta_2\) can be determined:
\[ \Theta_2(0) = \hat{T}_s \]

Suppose that the variation of the effective temperature is:
\[ \hat{T}_{rev} = \hat{T}_{rev} e^{j\omega t} \]

The surface temperature of the semi-infinite thick wall is:
\[ \frac{1}{j\omega C} + R \]
\[ \frac{1}{j\omega C} + R + \frac{1}{h} \hat{T}_{rev} e^{j\omega t} \]

with \( \frac{1}{\omega C} = R = \frac{d^*}{2\lambda} \) follows:
\[ \hat{T}_s = \frac{R (1 - j)}{R + \frac{1}{h} - jR} \hat{T}_{rev} e^{j\omega t} \]

The density of heat flow rate to the wall is:
\[ q_s = h(T_{rev} - T_s) = \frac{1}{hR + 1 - jhR} \]

The phase shift between the effective temperature and the density of heat flow rate then reads:
\[ \text{tg}(\varphi) = \frac{hR}{hR + 1} \]

So the angle is always smaller than 45°; only if \(h \rightarrow \infty\) the angle is 45° or 1/8 period.
3.3.4. Cyclic temperature variations at the surface of a slab

The mean density of heat flow rate in a very well insulated wall is small, at least much smaller than the instantaneous value resulting from e.g. a sudden solar irradiation. The mean heat flow into the wall is equal to the mean flow that leaves at the other side. So the heat storage of the wall for the passing heat flow is of minor significance. The storage for the heat flow from the interior to the wall and leaving at the same side is the real important one. This storage can have a significant influence on overheating in summer, the utilisation of passive solar energy in winter, the energy savings by nighttime setback and the cooling load. The indoor climate variations with a period of 24 hours are throughout the year the most dominant.

Suppose that the boundary conditions near a homogeneous wall (e.g. the brick inner leave of a cavity wall), are.

\[ x = 0 \quad T(0, t) = T_s e^{i\omega t} \]

\[ x = d \quad \left( \frac{\partial T}{\partial x} \right)_{x=d} = q_s = 0 \]

The solution can be found in a similar way as treated in section 3.3.2. We only give the result. It turns out that also here the thermal circuit of the heat flow and temperature at the surface is a thermal resistance and a heat capacitance in series:

\[ Z = \frac{T(0, t)}{q(0, t)} = \frac{d^*}{\lambda(1 + j)} \coth \left( (1 + j) \frac{d}{d^*} \right) = R + \frac{1}{j\omega C} \]

where: \(\coth\) is the hyperbolic cotangens.

With \( R = d_R / \lambda \) and \( C = \rho c d_c \) the solution reads:

\[ d_R = \frac{d^* \sinh (2d / d^*) - \sin (2d / d^*)}{2 \cosh (2d / d^*) - \cos (2d / d^*)} \]

\[ d_c = \frac{d^* \cosh (2d / d^*) - \cos (2d / d^*)}{\sinh (2d / d^*) + \sin (2d / d^*)} \]

And:

\[ \lim \frac{d_R}{d^*} = \frac{d}{3} \quad \lim \frac{d_c}{d^*} = d \quad \frac{d}{d^*} \to 0 \]

\[ \lim \frac{d_R}{d^*/2} = \frac{d}{d^*/2} \quad \lim \frac{d_c}{d^*/2} = d^*/2 \quad \frac{d}{d^*/2} \to \infty \]

The maximum value of \( d_c \) arises at \( d/d^* = \pi/2 \) and equals \( d_{c\text{max}} = 1.09 \). This thickness is often denoted as the optimal thickness. A reasonable approximation of the solution is:

\[ \frac{d}{d^*} < 1: \quad d_c = d \quad d_R = d/3 \]

\[ \frac{d}{d^*} \geq 1: \quad d_c = d^* \quad d_R = d^*/2 \]

(3.29)
The $d^*$ of stony materials (bricks etc) for 24-hour variations is approximate 10 to 13 cm. For the heat storage it makes not much sense to make a wall thicker than $d^*$ as the increase of the capacitance is small. Because wall thicknesses at the inside of the insulation are seldom larger than this 10 to 13 cm the heat capacitance can be estimated with the actual thickness. If one assumes equal climate conditions at either side of interior walls only half of their thickness adds to the heat capacitance of a room.

Eq.3.29 also gives a criterion for what can be considered as a ‘thick’ wall ($d > d^*$) and what as a thin one ($d < d^*$).

Investigate the difference between interior and exterior insulation with a sine function for the effective outdoor temperature and a constant indoor temperature. The insulation is situated between the concrete and a thin (reinforced) plaster-layer (no cavity).

Given: $R_i = 0.13 \text{ m}^2\text{K/W}$; $R_e = 0.04 \text{ m}^2\text{K/W}$; $t_0 = 24 \times 3600 \text{ sec}$. The other data can be found in the table above with some calculated values.

A. Exterior insulation

Performing this kind of calculations manually is very time-consuming. With the computer and software for symbolic algebra however it is a piece of cake.

The solution with the program MATHCAD is:
\[ \frac{\tilde{T}_b}{T_c} = 0.028 \Delta t = 4.2 \text{ hour} \quad \frac{\tilde{T}_{si}}{T_c} = 0.016 \Delta t = 6.12 \text{ hour} \]

where \( T_b \) = the temperature between insulation and concrete \\
\( T_{si} \) = the inside surface temperature

\[ \frac{\tilde{T}_b}{T_c} = 0.61 \Delta t = 4.07 \text{ hour} \quad \frac{\tilde{T}_{si}}{T_c} = 0.03 \Delta t = 5.06 \text{ hour} \]

It is noteworthy that \( \tilde{T}_{si} \) with exterior insulation is almost half the value with interior insulation and that the time lag is larger. Moreover the temperature drop over the concrete is much larger.

The relation between temperatures and heat flow densities at the both sides of a structure consisting of several layers is found by multiplication of the matrices of all layers.

- the determinant of the matrix of each layer is 1 and hence also the determinant of the matrix found after multiplication is 1.

- if the matrix equals \( \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) then the inverse is \( \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \)

- \( \tilde{q}(0) \) and \( \tilde{q}(d) \) are easily found:
\[ \tilde{q}(d) = \frac{\tilde{T}(0) - a_{11} \tilde{T}(d)}{a_{12}} \quad \text{and} \quad \tilde{q}(0) = \frac{-\tilde{T}(d) + a_{22} \tilde{T}(0)}{a_{12}} \]

sinh(x) = \( \frac{e^x - e^{-x}}{2} \) (hyperbolic sine)

\[ \cosh(x) = \frac{e^x + e^{-x}}{2} \] (hyperbolic cosine)

For a slab material the next relation between temperatures and heat flow densities at the surfaces can be deduced:

\[ \begin{pmatrix} \tilde{T}(0) \\ \tilde{q}(0) \end{pmatrix} = \begin{pmatrix} \cosh(1 + j)d \\ \frac{d^*}{\lambda(1 + j)} \sinh(1 + j)d \\ \frac{\lambda(1 + j)}{d^*} \sinh(1 + j)d \\ \cosh(1 + j)d \end{pmatrix} \begin{pmatrix} \tilde{T}(d) \\ \tilde{q}(d) \end{pmatrix} \]

\[ \tilde{q} = h (\tilde{T}_i - \tilde{T}_s) : \quad \begin{pmatrix} \tilde{T}_i \\ \tilde{q}_i \end{pmatrix} = \begin{pmatrix} 1 & 1/h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{T}_s \\ \tilde{q}_s \end{pmatrix} \]

The relation between temperatures and heat flow densities at the both sides of a structure consisting of several layers is found by multiplication of the matrices of all layers.

\[ \sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{(hyperbolic sine)} \]

\[ \cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{(hyperbolic cosine)} \]
3.3.5. Temperature fluctuations in a room

In the previous sections is treated that walls store heat at increasing temperature and release this heat at decreasing temperature. This storage stabilizes the indoor thermal climate: the rise of the indoor temperature is resisted because heat is stored in the material and is not available for the increase any more. Moreover, temperature decline is resisted by the released heat. Much storage results usually in a more comfortable indoor climate, it is however disadvantageous for energy saving by night setback.

Solar radiation entering a room (insolation) can be absorbed by material with much thermal capacitance (absorption by mass e.g. floors, walls) but can also be absorbed by materials with very little capacitance (lightweight objects e.g. furniture). The solar radiation absorbed directly by the mass is stored more effectively than the radiation that first increases the air temperature, because it is not stored at the surface where it is absorbed, and by the rise of the room air temperature is partially stored in the walls. The first case is denoted as primary storage and the second one as secondary storage. So the primary storage is important for a stable thermal indoor climate.

In figure above a simplified representation of the heat balance is depicted (a so-called first order model) to clarify the difference between primary and secondary storage. The symbols have the following meaning:

- $L_{tv}$ = steady state heat loss factor by heat transmission and by ventilation ($\Phi_{loss} = L_{tv}(T_i - T_e)$)
- $R_w, C_w$ = (mean) components of the wall impedance (see section 3.3.3) e.g.: $R_w = d/3\lambda$ and $C_w = \rho cd$
- $T_s$ = mean surface temperature
- $\Phi_{sunindir}$ = the variation around the mean solar radiation absorbed by the walls
- $\Phi_{sunindir}$ = the part that is released directly to the air
- $\tilde{T}_i$ = the variation of the indoor temperature
- $T_e$ = the outdoor temperature =constant.

The solution for this problem is:
\[ \bar{T}_i = \Phi_{\text{sundir}} \frac{1}{1 + \frac{R_i}{A_i}} - \frac{1}{L_{tv} A_i} + \Phi_{\text{sunind}} \frac{1}{L_{tv}} + \frac{1}{R_i + \frac{Z_w}{A_i}} \]

where: \( Z_w = R_w + \frac{1}{j\omega C_w} \)

With some algebra written in a more convenient way:

\[ \bar{T}_i = \frac{Z_w}{A_i / L_{tv} + R_i + Z_w} \left( \frac{\Phi_{\text{sundir}}}{L_{tv}} + \left( 1 + \frac{R_i}{Z_w} \right) \frac{\Phi_{\text{sunind}}}{L_{tv}} \right) \]

This solution shows that the primary storage is \((1+R_i/Z_w)\) time more effective than the secondary storage (usually \(R_i \gg Z_w\)). The result also shows that the magnitude of the surface coefficient is important: a small resistance results in a larger attenuation and less overheating.

Summarizing, for a stable thermal indoor climate important measures are:

- Much mass in a room. This mass should be assessable for the solar radiation, e.g. by a carpet on a concrete floor the storage is less assessable (additional thermal resistance). This conflicts with the requirement to realise a high interfacial temperature (section 3.2.2) for the feet.
- 'Lightweight’ dwellings and dwellings with interior insulation need good solar shading (exterior shading) and ventilation facilities as they can be very uncomfortable in summer.
- Mass with a high solar absorptivity and directly irradiated by the sun (primary storage). The absorptivity requirement often conflicts with the lighting e.g. a dark floor. In sunspaces (e.g. conservatories) sometimes barrels with water are directly exposed to the sun in order the get a more constant temperature and to benefit from the stored heat in the evening when it gets colder.

In the heating season the heat flow that enters a room by solar radiation can exceed the instantaneous heating demand. Mostly this happens in the early and late season. Then the heating demand becomes zero and the temperature rises above the desired set temperature. In order to prevent this uncomfortable overheating solar shading devices or extra ventilation will be used. Effectively this comes down in not using the potential solar energy. In a simple heat balance this is expressed by the solar utilisation factor that is smaller than one. If there is a lot of mass in the room (especially much primary storage) this utilisation factor will be higher than in the reverse case.

In a room that is used only just now and then there is no need for heating all of the time. If heating is switched off the structure will slowly cool down and the indoor temperature will decrease. If the system is switched on again later the same structure has to heat up and it will take some time before it is comfortable. So it is important to anticipate on this time lag and switch on some time before the planned
use of the room. This time is dependent on the desired temperature rise, the outdoor temperature, the heating system (air heating, radiator, floor), the maximum heat the system can deliver and obviously the thermal mass in the room. This is an important topic for the control of building HVAC systems.

It might be clear that for intermittent controlled rooms thermal mass has a negative effect on energy saving. More time is needed to heat up all the mass so the average indoor temperature will be higher. So in a lightweight timber frame house more energy can be saved by night setback than in a traditional brick house.

With the first order model also the effect of mass on night-time setback can be clarified.

The equation for the temperature of the thermal capacitance, $T_c$, (without solar radiation) is:

$$C_w \frac{dT_c}{dt} = \frac{T_e - T_c}{A_i / L_t v + R_i + R_w} = \frac{T_e - T_c}{R_i}$$

Suppose $T_e$ is constant and $T_i$ has the desired set temperature $T_{iset}$ between 8h and 24h. Consequently during that period also $T_c = T_{iset}$ . Starting from 24h the heating is switched off and the temperature decreases. At 8h the heating system is switched on with an infinite high heating power (in no time the temperature has the desired value again). The solution of the differential equation is:

$$T_c = T_e + (T_{iset} - T_e) \exp(-t / \tau) \quad \text{with} \quad \tau = R_i C_w$$

At 8 o’clock in the morning the temperature is:

$$T_{c8} = T_e + (T_{iset} - T_e) \exp(-8.3600 / \tau)$$

The heat needed to heat up to $T_{iset}$ is:

$$Q = C_w (T_{iset} - T_{c8}) = ((T_{iset} - T_e) / R_i) \tau \exp(-8.3600 / \tau)$$

This heat would be superfluous without heat capacitance ($\tau = 0 \Rightarrow Q = 0$). The larger $\tau$ the less the savings will be. Also insulation (larger $R_i$, so larger $\tau$ ) reduces the savings. In The Netherlands the influence of mass on the heating demand of dwellings is small. Mass is above all important for the summer (overheating).

### 3.4. Numerical methods

#### 3.4.1. Response factors

A condition to use response factors for solving a problem numerically is that the problem is linear, i.e. if the boundary condition is written as the sum of functions that are each a solution of the equation then the
complete solution equals the sum of the separate solutions (superposition criterion).

In order to clarify the use of response factors we introduce a problem with a known initial and boundary condition for the temperature $T(t)$ and for which the density of heat flow rate on a specific position on the boundary has to be calculated. We denote the temperature on time $t = n\Delta t$ with $n=1, 2, 3, \ldots$ and $\Delta t$ a fixed timestep, with $T(n\Delta t) = T_n$. Then the function $T=T(t)$ can be approximated with:

$$T(t) = \sum r(t-n\Delta t)T(n\Delta t) = \sum r(t-n\Delta t)T_n$$

Where $r(t)$ is a function with the shape of a triangle:

- $t < -\Delta t$ or $t > \Delta t$, $r(t)=0$
- $-\Delta t \leq t \leq 0$, $r(t)=1+t/\Delta t$
- $0 < t \leq \Delta t$, $r(t)=1-t/\Delta t$

If the heat flow $a(t)$ resulting from the unit triangular pulse $r(0)$ is approximated in the same way:

$$a(t) = \Sigma r(t-k\Delta t)u_k$$

then the total heat flow on time $n\Delta t$ turns out to be (superposition):

$$q(n\Delta t) = q_n = a_0T_n + a_1T_{n-1} + a_2T_{n-2} + \ldots$$

So at a specific time the density of heat flow rate is not only determined by the governing temperatures, but also by the history of the temperatures. The weighing factors are the response factors. The response factors need to be determined with the computer only once and the calculation of the heat flow for a time series of boundary temperatures is straightforward. This method is not limited to one-dimensional problems. The only condition is that it is a linear problem.

The number of response factors can be limited very much (e.g. 5) by also using the already calculated heat flow densities of previous times, e.g.:

$$q_n = a_0T_n + a_1T_{n-1} + a_2T_{n-2} + b_0q_{n-1} + b_1q_{n-2}$$

Advantages of the response factor method are:

- the calculation of the thermal behaviour for long time series (e.g. a year=365x24 hours) is very quick and accurate. In the past this was a strong argument but with the computers nowadays it is less important.
- For a 3D transient heat transfer problem the method is as quick as for a 1D one; only the calculation of the factors takes a little more time. This advantage is still relevant.
- The method is very general: with response factors also HVAC components can be characterized, whole rooms etc.

Drawbacks of the response factors method are:
- Only linear problems can be solved. If the surface heat transfer coefficient depends on temperature the method is not applicable.
- If one is interested in the temperature inside the construction then for each place separate response factors must be determined; so very many factors and the advantage disappears.
- Changing system components during the calculation can only be done if the factors for the new situation are known. So the method is little flexible.

Let \( a(t)=\exp(-bt) \) so \( a_0 = 1 \); \( a_1 = \exp(-b \Delta t) \); \( a_2 = (\exp(-b \Delta t))^2 \) etc. So \( a_n = a_1^n \).

So this is an infinite (geometric) series.

By definition yields \( q_n = a_0 T_n + a_1 T_{n-1} + a_2 T_{n-2} + \ldots \)
and also \( q_{n-1} = a_0 T_{n-1} + a_1 T_{n-2} + a_2 T_{n-3} + \ldots \).

With the formulas for \( a_n \): \( a_1 \ q_{n-1} = a_1 T_{n-1} + a_2 T_{n-2} + a_3 T_{n-3} \)

So \( q_n = a_0 T_n + a_1 \ q_{n-1} \)

By introducing the old density of heat flow rates in the series it turns out that only two factors are needed in this simple example to give the exact description of the exponential decrease. For the thermal problems in building physics the function \( a(t) \) often decreases exponentially for \( t \rightarrow \infty \) so the number of factors can be limited without loss of accuracy by using also the already calculated heat flow densities.

### 3.4.2. Finite differences

Just as for the numerical models of thermal bridges the first step is the spatial discretisation of the structure. If the problem is one-dimensional this means cutting the slab in a number of thin layers in such a way that the interface between materials coincides with the border of a layer. In the steady state case such a thin layer can be modelled with a thermal resistance whereas in the transient case a thermal capacitance halfway this layer is added. The equation (a balance equation) for one thin layer is:

\[
2 \lambda \frac{T_{j+1} - T_{j}}{\Delta x} + 2 \lambda \frac{T_{j-1} - T_{j}}{\Delta x} = \rho c \Delta x \frac{dT_j}{dt}
\]

This result can also be derived from the equation directly. Integrating the equation over the layer \( \Delta x \) we find:

\[
\rho c \frac{\partial}{\partial t} \int_x^{x+\Delta x} T \, dx = \int_x^{x+\Delta x} \lambda \frac{\partial}{\partial x} \frac{\partial}{\partial x} T = \lambda \left( \frac{\partial}{\partial x} T \right)_{x+\Delta x} - \lambda \left( \frac{\partial T}{\partial x} \right)_x
\]

If the layer is ‘sufficient’ thin the temperature in the layer changes linear with the place, so:

\[
\int_x^{x+\Delta x} T \, dx = T_j \Delta x \quad \text{en} \quad \lambda \left( \frac{\partial}{\partial x} T \right)_{x+\Delta x} = 2 \lambda \frac{T_{j+1} - T_j}{\Delta x}
\]
With this discretisation the thermal circuit of a wall is found by putting the layer circuit in series. (A cavity has negligible capacitance so is only a resistance).

To find a solution with the computer also the time needs to be discretised: $t_1, t_2, t_3, ..., t_k$ with intervals of $\Delta t$. Integration of the equation over $\Delta t$ yields:

$$\rho c \Delta x T_j(t_{k+1}) - \rho c \Delta x T_j(t_k) = \frac{2\Delta t \lambda}{\Delta x} (T_{j+1} - 2T_j + T_{j-1})$$

where $T_j$ = the mean value of $T_j$ over the interval $t_k$, $t_{k+1}$.

The mean value can be written as:

$$\bar{T}_j = \gamma T_j(t_{k+1}) + (1-\gamma) T_j(t_k)$$

$\gamma$ can have in principle all values between 0 and 1 e.g. if the temperature is assumed linear with time in the interval $\Delta t$ then $\gamma = 0.5$. $\gamma$ is called the implicitness degree.

**Fouriernumber**

With $a = \lambda/\rho c$ and the Fouriernumber $Fo = \frac{a \Delta t}{\Delta x^2}$ the equation reads:

$$T_j(t_{k+1}) - Fo \gamma \left(T_{j+1}(t_{k+1}) - 2T_{j+1}(t_{k+1}) + T_{j-1}(t_{k+1})\right) =$$

$$T_j(t_k) + Fo (1-\gamma) \left(T_{j+1}(t_k) - 2T_j(t_k) + T_{j-1}(t_k)\right)$$

**Cranck Nicolson**

The accuracy of the solution depends on the combination of the values for $\gamma$ and $Fo$. (see adjacent figure).

If $\gamma=0$ the method is called an explicit method: the value of the unknown $T_j(t_{k+1})$, is completely expressed in values of the previous time-step (backward discretisation). The method can be very instable dependent on the Fouriernumber. This is also an advantage, a wrong choice of the grid (wrong Fouriernumber) leads immediately to physically impossible results. The Fouriernumber for this method must always be less than 0.25 (see figure).

If $\gamma=0.5$ the way of discretisation is called the Crank-Nicolson method. The method is implicit as the value in the node $T_j$ cannot be expressed explicitly into already calculated values of the previous time-step. The solution for $t_{k+1}$ is found by solving the complete system of algebraic equations (tri-diagonal matrix). If the Fouriernumber is very large the result meanders a little bit, but the solution is always stable (doesn’t
That doesn’t mean that the result is also accurate. The choice of the Fourier number is a compromise between accuracy resulting from a fine spatial grid and instabilities caused by a too large Fourier number. A popular choice Fourier number $=1$, so $\Delta x^2 = a \Delta t$. In many simulations the time-step is fixed: the climate data are given as hourly values, so only the grid can be chosen.

If $\gamma=1$ the method is implicit with ‘forward’ discretisation. One could imagine that this is accurate when the temperature $T(t_{k+1})$ is reached very quick in the time-step, so with a very small Fourier number. As this is not efficient (fine grid) and this implicit method has no arithmetic advantages compared to the Crank-Nicolson method it is hardly used. An exception is the situation that a large Fourier number is unavoidable, e.g. a window-pane and a hourly time-step.

There are many numerical methods more sophisticated and accurate than the ones mentioned here and also more suited for multi-dimensional problems. Very popular is e.g. the finite elements. We only treated the simple methods in a very simplified way just to get some insight in the problems one can encounter.
Questions

The Fourier number is \( F_0 = \frac{a \Delta t}{\Delta x^2} \). A computer program is based on the Cranck-Nicolson discretisation scheme. Why should the spatial grid be chosen in such a way that \( F_0 >> 1 \)?

Describe what happens when two different materials with temperatures \( T_1 \) and \( T_2 \) respectively are brought in contact with each other. Distinguish between what happens initially and what after a long time. Assumed is that the materials have a finite thickness. Which material properties determine the interfacial temperature?

Which property determines the comfort class of floors and floor coverings and which material property?

A massive homogeneous wall is exposed at the exterior to a temperature variation that can be approximated with a sine function.
- Give the formula in a complex notation of this temperature
- Make a figure with the temperature in the wall as a function of the distance from one of the surfaces for different times. Show clearly attenuation and time lag

Give the formula for the impedance of a heat capacitance

Give a formula for the thermal resistance and the capacitance of a thin plate with a high thermal conductivity and insulated at the backside.

Give the 1st order equation of the plate mentioned above and the solution for a step change of the air temperature

What is attenuation depth? What determines the magnitude?

A wall is exposed to temperature variations at one side and a constant temperature at the other. Discuss the influence of this wall on the heat flow density at either sides of the wall.

Deep in the ground the shape of the temperature variation approximates much better a sine function than above the ground. Why?
4. Moisture transfer through porous materials

4.1. Introduction

The transfer of moisture through porous materials is an important topic of building science:

- **damage, durability**
  e.g. frost damage, damage by salts, moulds, moss, algae, discolouring, rot, paint peeling off.
- **drying processes**
  e.g. drying of wood.
- **indoor climate**
  air humidity, air quality, infiltration through joints that open by a varying humidity.
- **energy demand**
  e.g. heat of evaporation, moisture dependency of thermal properties of materials, indirect by use of extra ventilation to lower the air humidity, latent cooling load.

The theory of moisture transfer through assemblies is difficult. Research is still on the way. There are many rules of thumb and empirical formulas. In this chapter the emphasis is on the theoretical basis and understanding of phenomena.

Moisture transfer in materials is just like heat conduction a diffusion process. The equations and boundary conditions are similar:

- conservation of mass instead of conservation of energy;
- moisture flows are caused by gradients in the moisture potential instead of heat flows caused by temperature gradients.

There are also significant differences:

- The water vapour pressure has a maximal value (the saturation pressure) contrarily to temperature.
- The moisture content of a material has a maximum value while heat content has no limit at the relevant environmental conditions.
- Moisture diffusion in a material can occur in the form of water and of vapour. The potential for vapour is the vapour pressure and for water the capillary pressure or suction).
- Airflow can contribute significantly to vapour transfer in a porous material, much more than to the heat transfer.
- Water can evaporate resulting in 'vapour sources' ('sinks' for condensation).
- The relationship between heat-content and temperature is simpler than the relationship between moisture content and the moisture potential (non-linear). Moreover there is hysteresis i.e. the relationship depends on previous (in time) values of the moisture content.
- The transfer coefficients depend strongly on the moisture content: the equation is strongly non-linear.
- Temperature gradients influence much more the moisture transfer than moisture gradients influence the heat transfer. That doesn't mean that there is no influence: the thermal conductivity and specific heat capacity of a porous material depend on moisture content, at condensation heat is released. Thus heat and moisture transfer equations are linked.
- Gravity (hydrostatic pressure differences) can be important for water transport.
- The pore structure determines the moisture properties: two apparently similar bricks with equal densities and thermal conductivity might have completely different moisture properties.

To solve the moisture equations seems to be impossible without a computer. Yet for the understanding analytical approximations are indispensable. Analytical approximate solutions provide formulas for estimates that can help to value the computer solution.

### 4.2. The moisture transfer equation

#### 4.2.1. Capillary attraction (repulsion)

Many building materials are porous i.e. the material has a network of microscopic small pores or capillaries. Materials are isotropic if this network is in all directions alike. Material properties as heat conductivity, water vapour permeability and moisture permeability of anisotropic materials depend on the direction. An example of an anisotropic material is wood, where the bigger 'pores' are arranged in the direction of the fibres. We shall confine to isotropic materials; anisotropic materials are approached as isotropic.

A large porosity doesn't necessarily imply a high permeability for moisture: the pores can be closed. For bricks the baking degree is an important factor: during cindering a part of the pores get closed. Many insulation foams are hardly permeable for moisture as all pores are closed, e.g. PUR-foam. On the other hand UF-foam is very open. The volume of the pores of a dry material divided by the total volume (solid + air) is called the porosity. The volume fraction taken by the open pores is the open porosity, \( \psi_o \). A large porosity is usually paired with a low thermal conductivity and a low density. For example without pores the density of most inorganic materials is about 2645 kg/m\(^3\). Thus the density with pores is \( \rho = 2645(1 - \psi_p) \) where \( \psi_p \) is the porosity.
Moisture content and transport in porous materials are closely linked to the phenomenon that is called *capillary attraction*. Capillary attraction is caused by the *surface tension* of a liquid in a capillary. In a liquid a molecule is surrounded by other molecules that perform forces (cohesion). These forces outweigh each other and the resulting net force is zero. At the surface this holds no longer and there is a resulting force point towards the liquid and normal to the surface.

If the surface area is enlarged more molecules move to the surface at the cost of energy. The energy per m² surface area increase is called, somewhat misleadingly, the surface tension (coefficient) $\sigma$ (J/m²). The energy needed to increase the surface area along 1 meter $\Delta x$ is $E = \sigma l \Delta x$.

Because power = force x path or $E = F \Delta x$ it follows that $F = \sigma l$ (N).

The surface tension can be seen as the force per m² that acts at the edge of the surface trying to decrease the surface area. By this force a floating water droplet has an area as small as possible (hence the shape of a sphere if no other forces are present). The surface tension coefficient of pure water is $\sigma = (75.9 - 0.17 \Theta) \cdot 10^{-3}$ N/m ($\Theta$ in °C). So at a higher temperature the surface tension is lower. If there are salts dissolved in the water $\sigma$ increases.

If the surface of a fluid adjoins a plane wall also the molecules of the wall will act forces on the fluid (adhesion forces). If the adhesion exceeds the cohesion the fluid will be attracted by the wall (concave meniscus). In the reverse case the wall seems to reject the fluid (convex meniscus). Dependent on the fluid and the surface material a contact angle originates: $\theta$.

$\theta = 0^\circ$ much wetting, $\theta = 180^\circ$ no wetting.

The contact angle depends strongly on the contamination of the surface and dissolved material in the water. For example a greasy surface has a large contact angle; with soap in the water the contact angle is small. Many building materials have a contact angle close to 0°. Only for plastics the angle is usually larger.

Materials that have a contact angle with water smaller than 90° are called hydrophilic materials. If the angle is larger than 90° the material is hydrophobic.

If a capillary is placed at one end in water the surface between water and air in the capillary is either convex or concave. If it is concave (hydrophilic) the pressure in the water just below the surface is lower than the pressure of the surrounding air and the water is sucked into the capillary (much adhesion). If the surface is convex an external force is needed to get water into the capillary. The pressure difference across the surface is called the capillary suction and is larger as the contact angle is smaller and as the radius of the capillary is smaller.

$$ p_c = -\frac{2\sigma \cos \theta}{r} \tag{4.1} $$
where:  \( p_c \) = capillary suction \\
\( r \) = capillary radius

If the cross-section of the capillary is not circular the curvature radii of the meniscus appear in the formula for \( p_c \):

\[
p_c = -\sigma \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \cos \theta
\]

As a consequence of this pressure difference water molecules need a higher kinetic energy to escape than at a plane surface (=free surface) \( (r \to \infty, p_c = 0) \). If the fluid is in equilibrium with its environment the net transport of water vapour from the water surface to the environment is zero, or in other words the evaporation equals the condensation. Above a plane surface of water equilibrium is reached at 100% RH. Above a concave meniscus equilibrium is reached already below 100%. The formula for the lowering of the equilibrium water vapour pressure is called Kelvin’s law (or Thomson's law).

\[
p_v = p_{\text{sat}} \exp \left( \frac{p_c}{\rho_w R_v T} \right)
\]

where:  
\( p_v \) = equilibrium water vapour pressure \\
\( p_{\text{sat}} \) = saturation pressure \\
\( \rho_w \) = density of water \((1000\text{kg/m}^3)\) \\
\( R_v \) = gas constant for water vapour \((462\text{ J/kgK})\) \\
\( T \) = absolute temperature

From 4.2 follows a relation between the equilibrium relative humidity \( \varphi = p_v / p_{\text{sat}} \) and the capillary pressure that was given in eq.1.16.

For a given relative humidity equation 4.1 also gives the radius of the water filled capillaries that are in equilibrium with the environment. This is plotted in the adjacent figure.

\( e.g. \) at \( \varphi = p_v / p_{\text{sat}} = 0.98 \) \( p_c = -28 \times 10^5 \text{ Pa} \) \((28\text{ atmosphere!)}\) \( r \approx 0.05 \mu\text{m} \)

\[4.2.2. \text{ Moisture in materials}\]

In an equilibrium state (no evaporation or condensation) there is an amount of moisture in the pores of a hydrophilic material that depends on the relative humidity. Along with the increase of the relative humidity there is increase of water in the material. The main causes of this phenomenon are:

- **molecular absorption**

At the inside surface of pores there is a (mono-) layer of water molecules of which the thickness increases at increasing relative humidity. The resulting moisture content however is very small. At about 30% relative humidity the layer will be thick enough to close the smallest pores and a water meniscus arises.

- **capillary condensation**
In line with Kelvin's law the increasing relative humidity water condensates first in the small and after that in larger capillaries: \textit{capillary condensation}. The molecular absorption in the capillaries in which capillary condensation is not yet happening, is going on.

Until 98% relative humidity the moisture content is called the \textit{hygroscopic moisture content}. (Verify that at 98% the micro capillaries (r < 0.05 µm) are filled). Above 98% ones distinguishes yet the critical, the capillary and the maximal moisture content.

The \textit{maximal moisture content} is reached by submerging the material at vacuum. Then all open capillaries are filled with water. When a material is submerged under water and one waits until there is no perceptible change in the weight of this wet material, the reached moisture content is called the \textit{capillary moisture content}. This doesn't equal the maximal moisture content, as there is air-entrapment. The sensibility for frost damage of e.g. a brick usually increases as the difference between the capillary and the maximal moisture content is smaller (less air-entrancements).

The \textit{critical moisture content} is the lowest moisture content at which one supposes that there are still continuous water filled capillaries. Below the critical moisture content vapour transport dominates and above this water transport. The critical moisture content is larger than the maximal hygroscopic moisture content. In reality there is no clear breakpoint but rather a transition regime between vapour and water transport. The critical moisture content will be used above all for understanding what is happening. For the moisture models we use nowadays it is superfluous.

A material is dry if during drying at 105 °C the weight between two successive days changes less than 1%.

The moisture content of a material can be expressed in a volume percentage \( \psi \) (m³/m³ x100%), mass percentage \( u \) (kg/kg x100%) and in mass of moisture per volume material \( w \) [kg/m³]. Using the density of water (=1000 kg/m³) the relation between these three quantities is:

\[ \rho_{\text{mat}}u = 100w = 1000\psi \]

The plot of the moisture content in the hygroscopic region against the relative humidity is called the \textit{sorption curve}. One could expect that if the total pore volume of the capillaries with a radius smaller than e.g. \( r \) as a function of \( r \) is known, the moisture content as a function of the relative humidity can be derived with Kelvin's law. This appears not to be the case, mainly because \textit{air} is entrapped.
If the relative humidity is decreased step by step the moisture content at a specific relative humidity, often lays above the moisture content found with the measurement with increasing relative humidity. This phenomenon is called hysteresis. This is partially caused by the different ways capillaries can be filled at the same relative humidity. An example is the so-called ink-bottle-effect (see adjacent figure). Also experimental errors are easily made as it can take a very long time to reach equilibrium (sometimes weeks). So if one would wait longer the hysteresis might be less. In conclusion, the specific hygroscopic moisture capacity ($\xi$) and the specific capillary moisture-capacity ($\Xi$) (see Chapter 1, eq.1.18 and eq.1.21) appear to depend on the moisture content before a specific moisture content is reached and are not unambiguous functions of the relative humidity.

The sorption curve is determined by weighing the material at various values of the relative humidity. The measurement is time-consuming, as one has to wait for equilibrium each time the relative humidity is changed. Above ca 98% (above the hygroscopic regime) the curve increases enormously and an accurate measurement would require an extremely accurate measurement of the relative humidity. Unfortunately this cannot be done with the sensors available to date. Instead the moisture content can be measured for different capillary pressures with so-called pressure plates: $w = w(p_c)$.

If a material contains salt the material is more hydrophilic than without i.e. in equilibrium at a certain relative humidity (or capillary pressure) there is more moisture in the material.

Swelling or shrinking of porous materials is caused by changes of the capillary pressure in the pore water. According to Kelvin's law this pressure becomes less negative as the relative humidity increases. In other words the internal pressure increases also and causes swelling.

Wood is not isotropic and has very high hygroscopic moisture content. The main changes of the size occur in the direction tangential to the annual rings in the trunk. The changes in the radial direction are about $2/3$ of the tangential changes. The changes in the direction of the fibres (axial) are very small. When the relative humidity jumps from 60% to 50% wood shrinks in the tangential direction about 1%. This can be enough to break joints, to cause cracking of wood etc.

Also paper is very hygroscopic. At a low relative humidity paper can be folded less frequently before it tears than at a high relative humidity. Moreover for books yields that at a low relative humidity the glue gets brittle. A recommended value is around 50%.

At a varying relative humidity oil paint on canvas can break and get loose caused by the swelling and shrinking of the canvas. So a constant relative humidity is important in museums.

If a measurable physical effect of hygroscopic moisture is known as a function of the relative humidity this effect can reversibly be used to measure relative humidity: the so-called hygrometers. An example is the

<table>
<thead>
<tr>
<th>Influence of salt</th>
<th>Swelling</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wood</td>
<td>Paper</td>
</tr>
<tr>
<td>Canvas</td>
<td>Hygrometer</td>
</tr>
</tbody>
</table>
well-known hair-hygrometer that consists of a stretched hair of which the change of the length is a measure for the relative humidity: at an increase of R.H. = 40% to 60% the hair gets about 0.3% longer. In the thermo hygrograph this type of hygrometer is used. There are also hygrometers that use the change of electric properties (e.g. the capacitive hygrometer).

The moisture content of a material influences strongly the heat conduction, the specific heat capacity and the vapour resistance factor. So these properties are not constants as we assumed until now.

A simple correlation formula for moisture in materials is:

\[ w = w_c \frac{(C - 1)\varphi}{C - \varphi} \]

where: \( C = \) constant dependant on the material. This constant is determined with e.g. the moisture content at 80% R.H..

\( w_c = \) capillary moisture content

<table>
<thead>
<tr>
<th>a.</th>
<th>1 m³ concrete (( \rho_{mat} = 2300 \text{ kg/m}^3 )) has 100 kg of moisture per unity of volume. What magnitude has ( w, \psi ) and ( u )?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( w = 100 \text{ kg/m}^3, \ \psi = \frac{100}{2300} \times 100% = 4.3%, \ \psi = 10% )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>b.</th>
<th>Mineral wool (( \rho_{mat} = 12 \text{ kg/m}^3 )) has 100 kg moisture per unit of volume. What magnitude has ( w, \psi ) and ( u )?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( w = 100 \text{ kg/m}^3, \ \psi = \frac{100}{12} \times 100% = 833%, \ \psi = 10% )</td>
</tr>
</tbody>
</table>

For mineral wool '\( u \)' will usually be a large number and for this reason alone it is not popular. In case of wood '\( u \)' is preferred as this quantity doesn't change by shrinkage or swelling.

### 4.2.3. Vapour diffusion and capillary suction

When two different gasses with the same temperature and pressure are brought in contact with each other, they will mix: i.e. component 1 moves in the direction of component 2 and component 2 in the direction of 1. This phenomenon is caused by the thermal movement of the molecules and is called molecular diffusion.

At the indoor climate in buildings the partial pressure of water vapour is much smaller than the partial pressure of the dry air. Therefore for the diffusion of water vapour in air, the movement of the air is usually neglected and solely the mass flow of the vapour from a place of a high water vapour pressure to a place with a low water vapour pressure is considered. Fick's law is:
\[ \dot{g} = -\rho_s D \text{grad} \rho_v / \rho_s = -\delta_v \text{grad} \rho_v \]

where: 
- \( D \) = diffusion coefficient of water vapour in air
- \( \delta_v \) = vapour permeability coefficient of air: \( 1.8 \times 10^{-10} \text{s} \) or (At 5ºC)

For the formula with the water vapour pressure use is made of the ideal gas law (see Chapter 1) and a constant total pressure (1 atmosphere).

In a porous material the diffusion happens in the pores. The density of mass flow rate (Note: mass flow for 1 m² material and not 1 m² pore cross-section) is proportional to the pressure drop per meter material. The proportionality factor is \( \mu \) (>1), always less than the vapour permeability of air because:

- the surface area of the material is much larger than the area available for diffusion (the sum of the cross-section of the (open) pores). Materials with a high open porosity (\( \psi_o \)) have usually a small \( \mu \)-value. For many material yields that
  \[ 0.2 < \mu \psi_o < 0.6. \]

- the path taken by the vapour in the pores is much longer than the length of a straight line by the tortuosity of the pore-system.

The vapour resistance factor depends not only on the properties of the pore system but also on the moisture content. The main cause of this is the shortening of the diffusion path by water islands in the material (see also the influence of a temperature gradient). As a result of the islands the vapour resistance factor is smaller as the moisture content of the material is larger. By the dependency of moisture content on the relative humidity \( \mu \) can be written as a function of relative humidity. In simplified models the coefficient is often taken as a constant: the value for a low relative humidity (dry cup value). Besides Fick diffusion there are other phenomena that contribute to the transfer of ‘vapour’ in a porous material: surface diffusion (‘hopping’ of water molecules along the wall), Knudsen diffusion (pores with a diameter smaller than the mean free path of the air). Also a temperature gradient can contribute to diffusion: thermal diffusion. This contribution is very small and negligible for practice.

The coefficient for water transport is called the moisture permeability (see eq.1.20). The moisture permeability increases very much as the moisture content increases. An approximation of this coefficient with a moisture independent constant is very wrong.

Often one encounters a correlation formula like:

\[
\mu = \frac{\mu_0}{1 + a \phi^n}
\]

where
- \( \mu_0 \) = vapour resistance factor for the dry material
- \( a, n \) = material dependent constants

E.g. brick: \( \mu_0 = 15, a = 2.5 \) and \( n = 2 \).

Vapour resistance factor \( \mu \) (see eq.1.17).

Correlation formula for the vapour resistance factor
Suppose in a cylindrical capillary with a constant cross-section water flows solely by capillary suction. If the velocity is low and the flow laminar (Poiseuille flow) the mass flow is given by:

\[ g = -\rho_w \frac{r^2}{8\eta} \text{grad}(p_c) = -k_m \text{grad}(p_c) \]

\( \rho_w = 1000 \text{ kg/m}^3 \) (mass density of water)
\( \eta = \text{viscosity of water} \)
\( k_m = \text{moisture permeability} \)

In order to understand the moisture permeability of a complex capillary system we imagine a material that consists of parallel water-filled cylindrical pores. The mass flow \((A_g)\) in one capillary is:

\[ A_g = -\rho_w \frac{\pi r^4}{8\eta} \frac{\text{dp}_c}{\text{dx}} \]

Let the sum of the areas of the water filled capillaries in cross-section \(A\) be \(\Sigma \pi r^2\). In a slab with thickness \(d\) the amount of water is: \(\rho_w d \Sigma \pi r^2 \) kg. where \(w\) is the moisture content. This yields an expression for \(A\). So the formula for the density of moisture flow rate is:

\[ g = \frac{G}{A} = -\frac{w}{8\eta} \frac{\Sigma \pi r^2 \text{dp}_c}{\text{dx}} = -k_m \frac{\text{dp}_c}{\text{dx}} \]

This shows that coefficient \(k_m\) depends strongly on the moisture content because:
- the available cross-section for the flow is proportional to \(w\).
- the maximum radius of the water filled capillaries is larger at a higher moisture content.

4.2.4. The equation for the whole regime.

The moisture transfer equation combines equations for two extremes: the vapour transfer equation and the liquid water transfer equation. The experimental determination of the transport coefficients is reasonable nowadays: for vapour there is the traditional cup-method and for the capillary regime advanced non-destructive methods as NMR (nuclear magnetic resonance), \(\gamma\)-ray or neutrons. The transition regime (water and vapour transport have a comparable contribution to the total moisture transport) is difficult however. Usually the results of the vapour and water (capillary) regime are combined to one smooth curve.

In order to do this the equation must be written with the same potential. For isothermal transport there are three possible choices:

The relative humidity

\[ C_\phi \frac{\partial \phi}{\partial t} = \text{div} \ D_\phi \ \text{grad} \ \phi \]  

\( C_\phi = \xi \) or \( C_\phi = \Sigma \frac{\rho_w R_v T}{\phi} \)

where:

\[ D_\phi = \frac{\delta_a}{\mu} \rho_{\text{sat}} \] or \[ D_\phi = k_m \frac{\rho_w R_v T}{\phi} \]

relative humidity
In the transition regime the values are found from extrapolation of the values for vapour and water transport.

The capillary pressure:

$$C_{pc} \frac{\partial P_c}{\partial t} = \text{div} D_{pc} \text{ grad } P_c \tag{4.4}$$

where:

$$C_{pc} = \frac{\varphi}{\rho_w R_v T} \text{ or } C_{pc} = \Xi$$

And

$$D_\varphi = \frac{\delta}{\mu} p_{sat} \frac{\varphi}{\rho_w R_v T} \text{ of } D_\varphi = k_m$$

The moisture content:

$$\frac{\partial w}{\partial t} = \text{div} D_w \text{ grad } w \tag{4.5}$$

where:

$$D_w = \frac{\delta_a}{\mu} p_{sat} \frac{1}{\Xi} \text{ or } D_w = \frac{k_m}{\Xi}$$

The only potential that can be measured is the moisture content. So eq.4.5 is often preferable to eq.4.3 and eq.4.4. For example the moisture content can be derived from the weight increase (gravimetric method) but can also be determined with NMR. After \( w = w(x,t) \) is known \( D_w \) can be derived from eq.4.5 with known boundary conditions.

At the interface between two materials the density of moisture flow rate is continuous. Moreover one usually assumes that the vapour pressure and the capillary pressure are equal at both sides of the interface (see section 1.4.2). This is an approximation. If the materials have not an ideal contact an extra vapour resistance or hydraulic (for water) resistance is necessary. More complicated is the joint of mortar between bricks. The mortar joint is not homogeneous and the moisture properties are very difficult to determine. A simple approximation is to consider the joint as a resistance between two bricks.

The moisture content is not continuous at an interface: the capillary pressure is equal but in general not the corresponding moisture content (see figure). This is the main disadvantage of using the moisture content as moisture potential.

The moisture diffusivity, \( D_w \), depends strongly on the moisture content.
The moisture diffusivity, $D_w$, in the capillary regime is often represented with an exponential function. As in the hygroscopic regime $\xi$ is an increasing function of the moisture content and $\Xi$ a decreasing one, $D_w$ has a minimum near the critical moisture content.

### 4.2.5. The influence of the temperature

The gradient of vapour pressure is the main cause of vapour transfer. As a consequence there will be vapour transport when there is no gradient of relative humidity but a temperature gradient. Hence if the relative humidity is chosen as the potential a term with a temperature gradient has to be added:

$$
\bar{g} = -\frac{\delta \varphi}{\mu} \text{grad} \varphi - \frac{\delta \varphi}{\mu} \frac{d \varphi}{dT} \text{grad} T
$$

(4.6)

There are reasons to doubt the correctness of the formula above. The main reasons are (not thermo-diffusion as this is negligible):
- water transport (dependent solely on the relative humidity) in the hygroscopic regime
- surface diffusion (above all dependant on the relative humidity)

As the surface tension of the water-air interface in the pores is temperature dependent also a term with the temperature gradient will appear in the equation for water transport. There are hardly any measurements with a temperature gradient; most experimental work is done at isothermal conditions. The temperature term for water transport is usually supposed to be negligible and eq.4.6 is used for vapour transport.

**Eq.4.6 is a simplification. The vapour resistance factor depends on the relative humidity as a result of ‘short-circuiting’ of the relative humidity (Kelvin) by water-islands in the pores. In the isothermal case this means also a short-circuit for the vapour pressure. With a temperature gradient, however, there will be a temperature difference across the water islands, so also a vapour pressure difference across the islands. A first guess of the transport coefficient for the vapour flow caused by this extra pressure difference is the moisture dependent part of vapour permeability coefficient, as this dependency is caused by the short-circuiting of the relative humidity across the water-islands.**

With this assumption the equation for the density of mass flow becomes:

$$
\bar{g} = -\frac{\delta \varphi}{\mu} \text{grad} \varphi + \left(\frac{\delta \varphi}{\mu} - \frac{\delta \varphi}{\mu_0}\right) \varphi(\text{grad} \varphi)_{\text{island}}
$$

Where $\mu_0$ is the ‘dry cup’- value of $\mu$. If grad$p_r=0$ the vapour flows in the direction of the highest saturation pressure, so the lowest relative humidity. If the temperature gradient across an island (microscopic) equals the macroscopic gradient the equation above can be written as:

$$
\bar{g} = -\frac{\delta \varphi}{\mu} \text{grad} \varphi - \frac{\delta \varphi}{\mu_0} \varphi \frac{d \varphi}{dT} \text{grad} T
$$

In fact there is no reason to expect that the microscopic temperature gradient equals the macroscopic one. E.g. across an water island the temperature difference might be smaller than expected from the macroscopic gradient as the thermal conductivity of water is higher than of the air in the pores. Also the effect of surface diffusion for which the
relative humidity is the most probable potential will have an effect on the coefficients of the equation above.

The moisture content determines approximately up to which radius capillaries are filled. This means that:

$$p_c = \frac{\sigma(T)}{\sigma(T_0)} f(w)$$

where \(f(w)\) is the inverse function of \(w = w(p_c)\) at \(T = T_0\)

So with \(\sigma = (75.9 - 0.17\Theta) \cdot 10^{-3}\) (\(\Theta\) in °C):

$$dp_c = \frac{\sigma(T)}{\sigma(T_0)} \frac{df(w)}{dw} dw + \frac{f(w)}{\sigma(T_0)} \frac{d\sigma}{dT} dT \approx \frac{1}{\Xi} dw - \frac{0.17}{75} \frac{p_c}{dT}$$

So:

$$g = -k_m \frac{\Xi}{\Xi} \text{grad } w - \frac{k_m}{75} \text{grad } \frac{0.17 p_c}{\text{grad } T} - \frac{p_c}{0!}$$

So the flow caused by the temperature dependency of the surface tension moves towards the lowest temperature.

If \(p_c\) were used as potential the equation would read as:

$$\frac{C_{p_c}}{\Xi} \frac{\partial p_c}{\partial t} - \frac{C_{p_c}}{75} \frac{0.17}{\text{grad } T} \frac{\partial p_c}{\partial t} = \text{div } D_{p_c} \text{ grad } p_c$$

### 4.2.6. The influence of exterior pressure differences

Across a structure a pressure difference can arise by wind, air conditioning and mechanical ventilation and temperature differences. This results in convection through the pores of material and advection of water vapour. Usually this is very little and can be neglected but for very open materials like mineral wool this transport of vapour can exceed the transport by diffusion. Also by cracks in the material the convective vapour transport can be significant.

The pressure differences needed for water transport through a porous material are very large and are generally negligible compared with the capillary suction. Hydrostatic pressure can be of influence as e.g. 10cm water column is about 1000Pa. So the wind velocity must be higher than 40m/s to get a similar pressure difference (\(\Delta p = 0.5 \rho_v v^2\)).

E.g., mineral wool \(k_v \approx 29 \cdot 10^{-6}\) s, \(\mu = 1.2\)

Let \(p_v \approx 900\) Pa. With \(\delta_v = 1.8 \cdot 10^{-10}\) the vapour transport (eq. 1.25 and 1.28) is:

$$g = -\frac{1.8 \cdot 10^{-10}}{1.2} \text{grad } p_v - 0.62 \cdot 10^{-5} \cdot 29 \cdot 10^{-6} \cdot 900 \text{ grad } p$$

$$= -1.5 \cdot 10^{-11} \text{grad } p_v - 16182 \cdot 10^{-11} \text{ grad } p$$

Hence the contribution of convection equals diffusion if the total pressure difference is 1/1000 of the water vapour pressure difference across the material.
4.2.7. The influence of moisture on heat transport

In the preceding sections was shown that a temperature gradient can cause a moisture flow. Furthermore the saturation pressure and the different transport coefficients depend on temperature. So the moisture distribution in a material depends on the temperature distribution. The reverse is also true; the temperature distribution depends on the moisture distribution. The main cause is the evaporation and condensation of water vapour in the pores. For this phase change heat is needed or released. In the heat balance equation a heat sink (source) term arises:

\[
\rho c \frac{\partial T}{\partial t} = \text{div} \lambda \text{grad} T + L \text{div} \frac{\delta_\mu}{\mu} \text{grad} p_v
\]

(4.7)

where \(L=\) latent heat for evaporation (ca. 2500 kJ/kg).

In this equation there are some coefficients that depend on the moisture content:

\[
\rho c = \rho_0 c_0 + w_c \quad \text{and} \quad \lambda = \lambda_0 (1 + b w/\rho_0)
\]

where: \(c_w = \) the specific heat capacity of water (4.19 kJ/kgK)
\(b = \) a material-constant

Usually the influence of moisture on the heat transfer is small. Solely in case of moisture problems (e.g. wet insulation material) this will play a part.

4.3. Wetting and drying

4.3.1. A step change in the moisture content at the surface of a semi-infinite thick slab

If a dry porous volume of material is placed with one side in water the water will be sucked into the material. The increase of weight in time tells something about the suction capability of the material. Mathematically the boundary condition of this (sorption) experiment is a step change in the moisture content: from dry to the capillary moisture content. As but one material is considered the moisture content can be used as potential and the equation is:

\[
\frac{\partial}{\partial x} D_w \frac{\partial w}{\partial x} = \frac{\partial w}{\partial t}
\]

The initial and boundary conditions are:

\[
t < 0 \quad w = w_i \quad (\text{initial moisture content})
\]
\[
t \geq 0 \quad x = 0 \quad w = w_c \quad (\text{capillary moisture content})
\]
\[
x \to \infty \quad w = w_i \quad (\text{initial moisture content})
\]

The initial moisture content is the equilibrium moisture content at the relative humidity of the surrounding air. This is very small compared with the capillary moisture content (\(w_i \approx 0\)).

e.g. polyurethane: \(\rho_0 = 40 - 80\) kg/m\(^3\), \(\lambda_0 = 0.03\) W/mK, \(b = 0.4\)
The problem is similar to the problem of a temperature step (section 3.2.2). If the moisture diffusivity could be considered as a constant the solution would be the same as eq.3.5) where instead of ‘ΔT’ → ‘wc-wi’ and ‘a’ → ‘Dw’. The amount of stored moisture is similar to the amount of stored heat (eq.3.8) with ‘b’ → ‘√Dw’, so:

\[ \Delta m_w = A_s \frac{2 \sqrt{D_w}}{\sqrt{\pi}} (w_c - w_i) \sqrt{t} \] (4.8)

with \( A_s = \) the surface area

The weight increase by moisture uptake as a function of time can easily be measured by weighing. It can be proved that even if \( D_w \) is not a constant but an arbitrary function of the moisture content, the weight increase remains proportional to \( \sqrt{t} \). The proportionality constant is called the sorption coefficient (A). So:

\[ \Delta m_w = A_s A \sqrt{t} \] (4.9)

where: \( A_s = \) surface area in contact with water
\( A = \) sorption coefficient [kg/m²s¹/²]

For many materials the value of the sorption coefficient can be found in literature.

This sorption experiment is often used to determine the moisture diffusivity as a function of the moisture content. For that purpose the temporal and spatial distribution of the moisture content is measured (so the solution of this sorption problem is measured) and with the help of eq.4.9 \( D_w(w) \) is determined. For the non-destructive measurement of this moisture content sophisticated techniques are necessary: X-ray or neutrons or NMR.

The moisture diffusivity increases, as the material gets wetter. As a consequence the moisture gradient close to the surface is much smaller than far from the surface (less wet so a small coefficient, so diffusion is slower). This causes a front between wet and dry material. The position of the front, \( x_f \), can be written as:

\[ x_f = B \sqrt{t} \] (4.10)

where: \( B = \) the water penetration coefficient.

If the front is very steep, the water penetration coefficient can easily be estimated from the sorption coefficient:

\[
\frac{m_w}{A_s} = A \sqrt{t} \approx x_f (w_c - w_i) = B (w_c - w_i) \sqrt{t} \rightarrow B \approx A / (w_c - w_i)
\]

Generally a factor \( (\text{fac}B) \) has to be introduced:

\[ B = \text{fac}B \cdot A / (w_c - w_i) \] (4.11)
The sorption coefficient depends on the initial moisture content. In the adjacent figure experimental results for a particular material of \( A(\Psi) \) are plotted. Hence for an unambiguous determination of \( A \) the material must be dry.

Condensed water at the interface of two materials will be sucked into the material with the highest sorption coefficient.

---

**How much time is needed for wetting a brick of 10cm?**

\[ w_c = 290 \text{kg/m}^3, \quad A = 0.2 \text{ kg/m}^3 \cdot \text{s}^{1/2} \]

**Solution:**

\[ \text{volume} \times w_c = \text{area} \cdot A \sqrt{t} : \quad t = (145)^2 \text{ s} = 5.84 \text{ hours.} \]

So rather quick: 29 litres of water per m² in 5.84 hours

---

If a horizontal cylindrical capillary is brought into contact with water (at \( x = 0 \) and \( t = 0 \)) the water will flow into the capillary by suction.

At the entrance the suction pressure \( p_c = 0 \) and at meniscus (\( x = x_m \)) the pressure equals:

\[ p_c = -\frac{2\sigma \cos \theta}{r} \]

According to Poiseuille the mass flow rate for this laminar flow is:

\[ \rho_w \nu = -\rho_w \frac{r^2}{8\eta} \frac{dp_c}{dx} = \rho_w \frac{r^2}{8\eta} \frac{2\sigma \cos \theta}{rx_m} \]

Where \( \nu = \) the mean flow velocity and \( \eta \) the dynamic viscosity of water. The velocity can also be written as \( dx_m/dt \). So:

\[ \frac{dx_m^2}{dt} = \frac{\sigma r \cos \theta}{2\eta} \rightarrow x_m = \sqrt{\frac{\sigma r \cos \theta}{2\eta} \sqrt{t}} \rightarrow \nu = \frac{1}{2\sqrt{t}} \sqrt{\frac{\sigma r \cos \theta}{2\eta}} \]

The amount of water in the capillary is:

\[ m_w = A \rho_w x_m = \rho_w \pi r^2 \sqrt{\frac{\sigma r \cos \theta}{2\eta} \sqrt{t}} \]

From this formula one can conclude:

- Big pores have the mass flow velocity. So the big pores determine the position of the front between wet and dry material.
- The front moves proportional with \( \sqrt{t} \). The mass of water in the pores is also proportional to \( \sqrt{t} \) and is dominated by the big pores.
- Behind the front (in the wet part) the moisture content will still increase as it takes some time to fill the small capillaries.
Suppose there is a steep moisture front in the material at \( x = x_f \) and the moisture content decreases there from the critical moisture content \( w_{cr} \) till 0. Between \( x = 0 \) and \( x = x_f \) \( \frac{dw}{dx} \) is taken as constant. So:

\[ g_s = -D_{wc} \left( \frac{dw}{dx} \right)_0 = D_{wc} \frac{w_c - w_{cr}}{x_f} = \frac{A}{2\sqrt{t}} \quad \text{and} \quad D_{wc} = D_w(w_c) \]

\[
\Delta m = x_f \frac{w_c + w_{cr}}{2} = A \sqrt{t}
\]

After elimination of \( x_f \) \( A \) is:

\[ A = w_c \sqrt{D_{wc}} \sqrt{1 - \frac{w^2_{cr}}{w_c^2}} \quad \text{and} \quad B = \frac{2A}{w_c + w_{cr}} \]

With this formula too high values for \( A \) are obtained, i.e. instead of the critical moisture content higher moisture content should give better results.

The variables \( x \) and \( t \) of the diffusion equation with boundary conditions can be transformed into one variable: the so-called Boltzmann-transformation:

\[
\frac{d}{d\lambda} D_w \frac{d}{d\lambda} w = -\frac{\lambda}{2} \frac{dw}{d\lambda} \quad \text{with} \quad \lambda = \frac{x}{\sqrt{t}}
\]

Hence \( \Delta m / A \) is proportional to \( \sqrt{t} \):

\[
\frac{\Delta m}{A_s} = \sqrt{t} \int_0^\infty w(\lambda) d\lambda
\]

The diffusion equation can made dimensionless:

\[
w^* = \frac{w - w_e}{w_c - w_i}, \quad \lambda^* = \lambda \frac{w_c - w_i}{A}, \quad D_w^* = D_w \left( \frac{w_c - w_i}{A} \right)^2
\]

This yields:

\[
\frac{d}{d\lambda^*} D_w^* \frac{d}{d\lambda^*} w^* = -\frac{\lambda^*}{2} \frac{dw^*}{d\lambda^*} \quad \text{en} \quad \lambda^* = 0: \quad w^* = 1
\]

Moreover a constraint is:

\[
\int_0^\infty w^* d\lambda^* = 1
\]

If \( D_w^* \) is a constant one can prove that solely \( D_w^* = \pi/4 \) complies with the constraint, so \( D_w = (\pi/4)(A/(w_c-w_i))^2 \).

If \( D_w^* \) is an exponential function of \( w^* \) there will be one parameter that is free to choose and one determined by the constraint, e.g.:

\[
D_w^* = f(F) \exp(Fw^*)
\]

where \( f(F) \) is a function of the free parameter \( F \).
In a very good approximation the constraint is met if:

\[
f(F) = \frac{F^2 \exp(-F)}{2F - 1 - \left( F - 1 + \left( \frac{3}{2} - \frac{4}{\pi}\right) F^2 \right) \exp(-F)}
\]

The parameter \( F \) determines mainly the steepness of the wetting front and not at all the quantity of sucked water. The latter is determined by \( A \). For stony materials \( F \) is in the interval 5 to 10. So the moisture diffusivity increases tremendously with the moisture content. As a consequence there is a steep front between wet and dry material (at the front the coefficient collapses as a result of which the moisture transport also collapses).

Summarising the exponential moisture diffusivity can be written as:

\[
D_w = \left( \frac{A}{w_c - w_i} \right)^2 \frac{F^2}{2F - 1 - F - 1 + \left( \frac{3}{2} - \frac{4}{\pi}\right) F^2 \exp(-F)} \exp\left( \frac{F(w - w_c)}{w_c - w_i} \right)
\]

In the plot below the result of a computer calculation and the (Parlange-) approximation of \( f(F) \) are compared (\( D_{wc} = D_w(w_c) \)):

For example \( F=7.5 \) and \( w_i \ll w_c \):

\[
D_w = 2.2 \cdot 10^{-3} \left( \frac{A}{w_c} \right)^2 \exp\left( \frac{7.5}{w_c} \right)
\]

With the exponential approximation of \( D_w \) an estimate for the moisture penetration coefficient is:

\[
B \approx \left( 1 + \frac{1}{F \cdot 1.8} \right) \frac{A}{w_c - w_i}
\]
4.3.2. A step in the density of moisture transfer at the surface of a semi-infinite thick slab

Driving rain on a wall is a boundary condition for the density of moisture flow rate. The value depends mainly on the horizontal rain intensity and the wind speed near the surface. The prediction of the amount is still an important research topic. A sudden shower with a constant intensity is a step change at the boundary. Mathematically this is similar to the step change of the density of heat flow rate. Apart from the non-linearity of the diffusion equation for moisture transfer there is another important difference with the thermal problem viz the moisture content has a maximal value. Usually it is assumed that this is the saturation moisture content. If that value is reached not all rain will be absorbed and that part will run along the surface. If this happens often the surface will get dirty stripes where there is run-off. So it is interesting to know after how much time this will happen and how much time it takes to saturate a whole wall.

Suppose the wall is homogeneous and infinite thick. The equation with boundary condition for this problem is:

\[
\frac{\partial}{\partial t} w = \frac{\partial}{\partial x} D_w \frac{\partial}{\partial x} w
\]

with b.c. \( t < 0 \quad w = w_i \)

\( x = 0 \quad t \geq 0 \quad g_s = -D_w \frac{\partial}{\partial x} w \)

\( x \to \infty \quad t \geq 0 \quad w = w_i \)

If \( D_w \) is constant, this problem is fully similar to the thermal problem with an increase of the density of heat flow rate. (section 3.2.3). So also the solution is similar (eq.3.10). The rise of the moisture content near the surface is given by formula 3.11 (\( q_s \rightarrow g_s \), \( a \rightarrow D_w \), \( \Delta T \rightarrow \Delta w \)):

\[
\Delta w_s = \frac{2\sqrt{t}}{\sqrt{\pi D_w}} g_s
\]

(4.12)

With formula 4.8 and 4.9 \( D_w \) can exchanged by the water absorption coefficient:

\[
\Delta w_s = \frac{4}{\pi} \frac{w_c - w_i}{A} g_s \sqrt{t}
\]

So the time needed to reach the capillary moisture content at the surface \( (\Delta w_s = w_c - w_i) \) is for a constant \( D_w \):

\[
t_1 = \left( \frac{\pi}{4} \right)^2 \left( \frac{\Delta A}{A} \right)^2 = 0.62 \left( \frac{\Delta A}{g_s} \right)^2
\]

This formula gives a high estimate as the moisture will stay closer to the surface due to the decrease of \( D_w \) in the dryer part of the wall (there is a moisture front). Suppose that at \( t = t_1 \) the moisture distribution is equal to
the moisture distribution with boundary condition \( w=w_c \) (water sorption) at \( t=t_0 \) \((t_0 \neq t_1)\). Then the mass increase for \( t>t_1 \) and the density of moisture flow rate would be (eq.4.9):

\[
\Delta m = A\sqrt{t-t_1+t_0} \quad \text{and} \quad g = \frac{1}{A_s} \frac{dm}{dt} = \frac{A}{2\sqrt{t-t_1+t_0}}
\]

This means that with this assumption for \( t>t_1 \) the suction flow rate is inversely proportional to moisture content in the material as:

\[
g \frac{\Delta m}{A_s} = \frac{A^2}{2} \quad (4.13)
\]

If this result is used at \( t=t_1 \): \( g=g_s \) and \( \Delta m = g_s A_s t_1 \) one finds:

\[
t_1 = 0.5 \left( \frac{A}{g_s} \right)^2
\]

More general: \( t_1 = f a c g_s \left( \frac{A}{g_s} \right)^2 \) (4.14)

One can prove that independent of \( D_w \): \( 0.5 < fac g_s < 0.62 \)

A condition for formula 4.14 is that on the moment of dripping a front exists in the material i.e. the water content of a saturated wall must be larger than the water that entered the wall, so \( dA_s(w_c-w_i)>A_s g_s t_1 \) or with eq.4.14:

\[
(g_s d) > fac g_s \frac{A^2}{w_c-w_i} \quad (4.15)
\]

For a front the wall must be sufficient thick or the density of moisture flow rate must be high. If this is low or the wall is thin there will hardly be a gradient and consequently no front. The time it takes for dripping is:

\[
t_1 = \frac{(w_c-w_i) d}{g_s} \quad (4.16)
\]

After that time the wall is also completely wet.

Capillary forces are not responsible for rain penetration. An extra force is needed for this. For example gravity can be the cause: water goes down in the wall and leaves at a lower place. In practice this is not the explanation: the flow resistance of bricks and mortar is such large that only little water will leave the wall. Another force can be the wind pressure. Also this pressure in combination with the large flow resistance will not result in significant water flow on the surface.
Research proved that the main cause is cracking. E.g. not completely filled vertical joints of brickwork are paths for rain penetration.

**How long does it take till dripping occurs on a brick wall if:** the driving rain intensity is 10mm/hour, \( d = 0.1 \text{m} \), \( A = 0.2 \text{ kg/m}^2\text{s}^{1/2} \), \( w_c = 290\text{kg/m}^3 \)?

The density of moisture flow rate is: \( g_s = 10/3600 \text{ kg/m}^2 \). The condition given by eq.4.15 is fulfilled as:

\[
g_s d = 1/3600 < 0.62 \text{ or } 0.04/290
\]

The elapsed time till the moment of dripping is

\[
0.5(0.2 \cdot 360)^2 < t_1 < 0.62 (0.2 \cdot 360)^2 \text{ or } 2600 < t_1 < 3200\text{sec}
\]

So even with this very high driving rain intensity it takes very much time i.e. for brickwork dripping is a rare phenomenon.

If \( D_w \) is an exponential function of the moisture content (section 4.3.1) one can find with a computer:

\[
\text{fac}_{g_s} = 0.5 \left( 1 + \frac{0.24}{1 + 0.4F} \right)
\]

With the assumption that the moisture distribution is very similar to the distribution resulting from a constant moisture content at the surface (section 4.3) the place \( x_f \) of the moisture front for \( t > t_1 \) can be estimated. Suppose the distribution at \( t_1 \) equals the distribution of water sorption at \( t = 0 \), or

\[
g_s t_1 = A/\sqrt{t_0} \quad \Rightarrow \quad t_0 = \left( \text{fac}_{g_s} - \text{fac}_{g_s}^2 \right) A^2/g_s^2 = 0.25A^2/g_s^2
\]

With \( x_f = B/\sqrt{t} \)

4.3.3. Drying

The transfer of moisture in a drying material usually is caused by capillary transport and diffusion at the same time and from the surface to the environment convection is the main mechanism. Especially the slow transport dominates drying i.e. the molecular diffusion in the material and the convection at the boundary. We know from experience that we can get wet in a short time but that drying takes much longer.

The accuracy of the calculation of the moisture content in a material subjected to a cyclic wetting and drying situation depends much more on the accuracy drying can be predicted than on the wetting part e.g. driving rain is usually completely absorbed by a brick wall and the final result is not very sensitive for the exact form of the liquid water diffusivity. So the description of \( D_w \) with an exponential function cannot be used, as this is only valid for capillary transport.

In order to understand what happens it is important to know the moisture content that marks the transition from vapour diffusion to capillary transport. This is called the critical moisture content. In reality such a sharp transition doesn’t exist and there is rather a small region
where both transport phenomena occur at the same time. In the simplified theory two drying stages are usually distinguished:

I  Constant drying

In this stage the moisture content is still so high that the gradient in the moisture content can be neglected (large moisture diffusivity for liquid water transport) and the vapour pressure at the surface equals the saturation pressure. The drying rate is then similar to the evaporation of free water. This yields as long as the moisture content is higher than the critical moisture content.

Suppose a box-shaped material with a capillary moisture content starts to dry at $t = 0$ at only one side, so one-dimensional, i.e. all sides but one are sealed water and vapour tight. The drying rate (the density of moisture flow rate at the surface) is $g_s$ and is constant in this stage.

At the end of the first drying stage the material will have the critical moisture content provided that the drying rate is very low ($d/g_s$ is small) and the material not very thick. The drying time is approximate:

$$t_i = \frac{\text{d}W_c - W_{cr}}{g_s} \text{ en } g_s = \beta (p_{sat} - p_s)$$ (4.17)

where: $\beta$ = the surface coefficient for vapour transfer

II  Decreasing drying rate

In this stage the surface is so dry that the vapour pressure decreases as the drying gets on and consequently the drying rate. One can imagine a drying front in the material at which the moisture content is critical and between that front and the surface there is vapour diffusion. Especially stage II takes very much time as diffusion is slow

As heat is needed for evaporation the temperature at the drying front will be lower and also the saturation pressure. This slows down the drying. If radiation and conduction heat flows to the surface are negligible the minimum surface temperature one can get equals the wet-bulb temperature. Usually the temperature is much closer to the air temperature and the thermal effect on the total drying time is very small.

Suppose the drying front is at $x = x_f$. Then:

$x > x_f$  $w = w_{cr}$
$x < x_f$  $w = w_h$ (= moisture content at equilibrium with its environment)

The density of vapour flow rate is:

$$g_s = \beta (p_s - p_v) = \frac{\delta_a}{\mu x_f} (p_{sat} - p_s) \rightarrow g_s = \frac{p_{sat} - p_s}{\beta} + \frac{\mu x_f}{\delta_a}$$
At first \(x_f=0\) the drying rate is \(g_s = g_0\). When the drying front reaches the backside \(x_f=d\) the drying rate is: \(g = g_0(1+d\mu/\delta_a)\), where \(g_0 = \beta(p_{sat} - p_e)\)

So the mean drying rate is reached when the front is at \(x_f = d/2\) and consequently the drying time is:

\[
t = \frac{(w_{cr} - w_h)d}{g_s} = \frac{(w_{cr} - w_h)d}{g_0} \left(1 + \frac{d\mu\beta}{2\delta_a}\right) \approx \frac{(w_{cr} - w_h)}{2(p_{sat} - p_e)} \frac{d^2\mu}{\delta_a} (4.18)
\]

The 2nd drying time is proportional to the square of the thickness. The surface coefficient \(\beta\), and so the wind, has no influence on this time. Solely with \(p_{sat}\) and \(p_e\) the drying time can be shortened. Especially \(p_{sat}\) is important as \(p_{sat}\) increases very much with the temperature, e.g. at 100°C \(p_{sat} = 10^5\)Pa.

The graph of the drying rate \(dm/dt (=Asgs)\) as a function of the mass of a drying material is bent when the critical moisture content is reached at the surface, viz the drying is slower afterwards. By repeating the measurement of \(m = m(t)\) for different drying rates this bend can be extrapolated to the situation of very slow drying (the moisture content is everywhere the same in the material) and the critical moisture content can be determined.
How long is the first drying time of 10 cm brick, if: \( \Theta_e = 15^\circ C \), R.H. = 70%, \( \beta = 15 \cdot 10^{-8} \) s/m?

\[ w_c = 290 \text{ kg/m}^3, \ w_{cr} = 100 \text{ kg/m}^3, \ d = 0.1 \text{ m} \]

\[ g_s = \beta (p_{sat} - p_e) = 15 \cdot 10^{-8} \cdot (1706 - 1194) = 77 \cdot 10^{-6} \text{ kg/m}^2 \text{s} \]

\[ t_1 \approx \frac{0.1}{77 \cdot 10^{-6} (290 - 100)} = 0.25 \cdot 10^{-6} \text{ sec} = 2.8 \text{ da ys} \]

How long is the second drying time if \( \mu = 7.55 \) and \( w_h = 5 \text{ kg/m}^3 \)?

\[ \delta_a = \frac{1.8 \cdot 10^{-10}}{7.55 \cdot 15 \cdot 10^{-8}} = 1.59 \cdot 10^{-4} \text{ m} \]

\[ t_2 = \frac{100 \cdot 5}{77 \cdot 10^{-6} \cdot 0.1 (1 + \frac{1}{2} \cdot 0.1 \cdot 1.59 \cdot 10^{-4})} = 39 \cdot 10^6 \text{ sec} \approx 1.2 \text{ year} \]

So it takes very long before a material is completely dry. The second drying time is decisive. Show that the drying time is about 2.3 days at 100°C and the same p_e.

For the drying time of a wall with drying at both sides one sometimes encounters the formula: \( t = s d^2 \cdot 10^4 \text{ (days)} \) where for brick \( s = 0.3 \) and for other stony materials \( s = 1.2 \) or higher. For drying at but one side the formula is \( t = 4s d^2 \cdot 10^4 \). Eq.4.18 yields:

\[ s = \frac{w_{cr} \cdot w_h \cdot \mu \cdot 10^{-4}}{2(p_{sat} - p_e) \delta_a} \cdot \frac{1}{3600 \cdot 24} \]

With the values of the example above \( s = 1.12 \). It is clear that \( s \) cannot be a fixed number but depends heavily on \( p_{sat} \). The proportionality with \( d^2 \), however, is generally true as one can proof by making the differential equation dimensionless.

The theory above is very simplified:

- If the surface coefficient of vapour transfer is very large e.g. by blowing, the first drying phase will be reached much quicker and the critical moisture content will be reached at the surface while deeper in the material the moisture content is higher. As a consequence the second drying phase will start different.

- The moisture diffusivity changes gradually with the moisture content and the moisture content at the back doesn’t remain critical till the front is there but decreases also. As a result the front position is rather proportional to ‘\( t/d \)’ (see figure) instead of with \( ‘\sqrt{t}’ \).
The amount of water that evaporates per m² per second is:

\[ g = (w_{cr} - w_h) \frac{\Delta x_f}{\Delta t} \]

This yields the differential equation below:

\[ g = (w_{cr} - w_h) \frac{dx_f}{dt} = \frac{p_{sat} - p_e}{\beta + \mu x_f / \delta_a} \]

or:

\[ dt = \frac{w_{cr} - w_h}{g_s} dx_f (1 + x_f \mu \beta / \delta_a) \]

\[ t = \int_0^{x_f} \frac{w_{cr} - w_h}{g_s} dx_f (1 + x_f \mu \beta / \delta_a) = \frac{w_{cr} - w_h}{g_s} x_f (1 + x_f \mu \beta / 2 \delta_a) \]

### 4.4. Rising damp

If a wall is standing in water moisture will rise in the wall by capillary suction. The moisture content as a function of height obviously depends on the capillary properties of the wall (e.g. the water sorption coefficient) but also on the evaporation of water at the surface. By evaporation the moisture content of the surface is, just as by drying, more dry than inside the wall. To solve this 2-dimensional problem with capillary and vapour transport one needs a computer. Here we treat some simplified solutions, which show the influence of boundary conditions and material properties on the moist height.
4.4.1. Vertical moisture distribution without evaporation

In section 4.3.1, the capillary suction in a horizontal capillary was treated. In an infinite long horizontal capillary the flow will never stop, but the velocity only decreases more and more (by \(1/\sqrt{t}\)). In a vertical capillary however, the flow will stop by gravity. If the water in the capillary rises till height \(h\) above the free surface of the water the hydrostatic pressure in the capillary at that height is: \(-\rho_wgh\) where \(g\) is the acceleration of the gravity. In equilibrium this pressure equals the capillary suction. So:

\[
p_c = -\rho_wgh = -\frac{2\sigma \cos \theta}{\rho} \Rightarrow h = -\frac{2\sigma \cos \theta}{\rho g}
\]  (4.19)

In narrow capillaries the water rises much higher than in wide ones. The suction pressure in the capillary is \(p_c(x) = -\rho_wgx\), so independent of the capillary geometry. In a wall the capillary suction is the same as the moisture content at \(x\) is the moisture content that belongs to \(p_c = -\rho_wgx\). If the wall consists of several layers that have an ideal hygric contact then this has no effect on \(p_c = p_c(x)\). The moisture content in each layer at each height is given by the relation \(w = w(p_c)\) of the concerned material.

The time need for saturating a dry 10 cm thick brick wall is of the order of hours (see section 4.3.1). So for one meter this is 100 times longer or of the order of a week. For rising dampness the gravity will slow down this wetting and the rising time is longer than a week. If there are horizontal joints with a high hydraulic resistance the time will increase again.
To what height will water rise in a vertical circular capillary with \( r = 1 \mu \text{m} \)? (\( \theta = 0 \) and there is no evaporation)

This is equilibrium if \( p_c + \rho_w g h = 0 \), where \( h \) = vertical height and \( g \) = acceleration due to gravity. So

\[
h = \frac{2\sigma}{\rho_w g r} = \frac{2.75 \cdot 10^{-3}}{1000 \cdot 9.8 \cdot 10^{-6}} = 15 \text{m}
\]

At what height the critical moisture content is reached if the relative humidity at the critical moisture content is 99.7%? Evaporation can be neglected

\[
x = 0: p_c = 0, w = w_c \quad \text{and} \quad x = h: \phi = 0.997, w = w_{cr}
\]

With Kelvin:

\[
p_c = \rho_w R T \ln(\phi) = 1000 \cdot 462 \cdot 300 \cdot \ln(0.997) \approx -4.2 \cdot 10^5 \text{ Pa}
\]

This moisture content is reached at:

\[
h = \frac{p_c}{\rho_w g} = 43 \text{ m}
\]

### 4.4.2. Rising damp with drying

In reality evaporation can seldom be neglected and the wall will loose moisture that also in a steady state situation has to be supplied by a vertical water flow at the bottom. Due to this flow there is a pressure drop and the moisture will rise less high than in section 4.4.1 in the wall. Not the gravity but the evaporation turns out to be the most important for the height of rising damp.

Suppose:
- horizontal gradients in the wall and the influence of gravity can be neglected
- there is evaporation \( g_e \) (kg/m²/s) only at one side and independent of the height \( x \) above the water table.

If it is wet until height \( h \) the evaporation is \( h g_e \) (kg/s.width). Suppose above \( h \) the material is dry and below \( h \) wet i.e. the vertical moisture flow at height \( h \) can be neglected, then the density of moisture flow rate at the bottom is:

\[
g_0 = h g_e / d \quad \text{with } d = \text{thickness of the wall}
\]

To get an estimate of the height we use the eq.4.13 and approximate the mass of water with \( A_n h_w c \)

\[
g_0 \Delta m / A_s = A_n^2 / 2 \Rightarrow \frac{h g_e}{d} h w c = A_n^2 / 2 \Rightarrow h = 0.7 A_n \sqrt{d / g_e w_c}
\]

With a more accurate approximation the height is found to be:
Above the height $h$ the humidity at the surface is less. For moulds the situation of a completely wet surface appears to be less favourable than just above the height $h$. So not surprisingly the height of rising dampness can be seen by a dirty band of moulds.

If the moisture permeability of the joints is much lower than the resistance of bricks ($k_m$ and also $A$ is smaller) the joints determine the rising height in brickwork and the wet part of a wall ends at a joint.

If the wall is plastered with a plaster of low moisture permeability moisture will not rise very high in the plaster, but inside the wall it will rise much higher as there will be less evaporation. If there are salts in the water these will be transported to the interface of wall and plaster. Increasing salt concentration can cause crystallisation and the plaster might come loose.

**Example**

How high will water rise with evaporation at one side if: $T_e = 15^\circ C$, R.H. = 70%, $\beta = 15 \cdot 10^{-8}$ s/m, $w_c = 290$ kg/m$^3$, $d = 0.05$ m and $A = 0.2$ kg/m$^2$s$^{1/2}$?

$$g_e = \beta (P_{sat} - P_e) = 15 \cdot 10^{-8} \cdot (1706 - 1194) = 77 \cdot 10^{-6} \text{ kg/m}^2\text{s}$$

The rising height is $h = 0.2 \sqrt{\frac{0.05}{290 \cdot 77 \cdot 10^{-6}}} = 0.3$ m

This example shows that evaporation and not the gravity is determines the height.

**Per horizontal linear metre the evaporation in a band** $x, x + \Delta x$ is: $g_e \Delta x$.

This must be equal to the difference between the moisture flow rate in the wand at $x$ and at $x + \Delta x$ or:

$$g_e \Delta x = d \Delta \left( D_w \frac{dx}{dx} w \right) \quad \text{or:} \quad \frac{dx}{dx} D_w \frac{dx}{dx} w = \frac{g_e}{d}$$

with boundary condition: $x = 0 \quad w = w_c$

The evaporation depends on the vapour pressure near the surface. As long as the moisture content is higher than critical the saturation pressure can be used and the evaporation is constant. This means:

$$D_w \frac{dx}{dx} w = \frac{g_e}{d} x dx + \left( D_w \frac{dx}{dx} w \right)_{x=0}$$

$$\int_{w_c}^{w} D_w (w) dw = \frac{1}{2} \frac{g_e}{d} x^2 + \left( D_w \frac{dx}{dx} w \right)_{x=0}$$
Suppose at \( x = h \) the moisture content is critical. For \( x > h \) there is mainly diffusion and the density of moisture flow rate is small. If this is assumed to be zero then:

\[
\left( D_w \frac{d}{dx} w \right)_{x=h} = 0 \quad \text{or} \quad \left( D_w \frac{d}{dx} w \right)_{x=0} = -\frac{g_c}{d} h
\]

so

\[
\int_{w_c}^w D_w (w) dw = \frac{1}{2} \frac{g_c}{d} x^2 - \frac{g_c}{d} x h
\]

From this and a known \( D_w \) the moisture content \( w = w(x) \) can be determined. If the wall consists of but one material the integral can be estimated:

If \( D_w = \left( \frac{A}{w_c} \right)^2 D_w^* (w) \)

then for the exponential approximation of \( D_w \) the equation is:

\[
- \left( \frac{A}{w_c} \right)^2 \frac{D_w^* (w_c) - D_w^* (w)}{F} = \frac{g_c}{d} \left( \frac{1}{2} x^2 - x h \right)
\]

So if at \( x = h \), \( w = w_c \) \( h \) equals:

\[
h^2 = \frac{2d}{g_c} A^2 \frac{D_w^* (w_c) - D_w^* (w)}{Fw_c} \rightarrow h \approx A \frac{d}{\sqrt{w_c g_c}}
\]

### 4.5. Vapour transport

#### 4.5.1. Similarity between vapour and heat transport

Under isothermal conditions the saturation vapour pressure is constant and the diffusion equation is:

\[
\frac{\xi}{p_{sat}} \frac{\partial p_v}{\partial t} - \frac{\partial}{\partial x} \frac{\delta p_v}{\mu} = \frac{\partial p_v}{\partial x}
\]

If in this equation \( p_v \) is exchanged by \( T \), \( \delta p_v / \lambda \) and \( \xi / p_{sat} \) by \( \rho c \) one gets the equation for transient heat conduction (chapter 3). So for constant coefficients all solutions treated in chapter 3 can be translated to solutions of the vapour diffusion equation. Analogous to a thermal effusivity (section 3.2.2) a water vapour effusivity \( b_v \) [s^{3/2}/m] and analogous to the thermal diffusivity (section 3.1) a water vapour diffusivity \( a_v \) [m^2/s] can be introduced.

\[
b_v = \sqrt{\frac{\delta p_{sat}}{\mu p_{sat}}} \quad \text{and} \quad a_v = \frac{\delta p_{sat}}{\mu \xi}
\]

(4.21)

Note that \( a_v = D_w \) (eq.4.5) in the hygroscopic regime.
The coefficients are not constant but depend on the relative humidity and the temperature. Within the range $0.2 < \varphi < 0.8$ the assumption of constant coefficients is reasonable. This range is sufficient long for the relative humidity’s that occur indoors.

Another important difference is the value of the vapour diffusivity $a_v$. Usually this is a factor 100 smaller than the value for heat. So the penetration depth (see section 3.2.2 eq.3.21) for sinusoidal variations is about a factor 10 smaller than for temperature variations with the same period. This means that the daily variation of the air humidity affects only a thin layer of the material, e.g. 1 cm. Often this is the plaster, floor covering etc. Behind parts with a vapourtight paint there is hardly any variation ($\mu \rightarrow \infty$).

**Brick**

\[
\begin{align*}
\alpha &= \frac{\lambda}{\rho c} = 0.55 \div (1500 \div 840) = 44 \cdot 10^{-8} \text{ m}^2/\text{s} \\
\mu &= 7.55 \quad \xi = 13 \text{ kg/m}^3 \quad p_{\text{sat}} = 2340 \text{ Pa} \\
a_v &= (1.8 \cdot 2340 \cdot 10^{-10}) \div (7.55 \cdot 13) = 43 \cdot 10^{-10} \text{ m}^2/\text{s}
\end{align*}
\]

**Concrete**

\[
\begin{align*}
\alpha &= \frac{\lambda}{\rho c} = 1.9 \div (2500 \div 840) = 90 \cdot 10^{-8} \text{ m}^2/\text{s} \\
\mu &= 100 \quad \xi = 40 \text{ kg/m}^3 \quad p_{\text{sat}} = 2340 \text{ Pa} \\
a_v &= 105 \cdot 10^{-12} \text{ m}^2/\text{s}
\end{align*}
\]

The R-C circuit of an infinite thick wall is fully similar to the thermal circuit (eq.3.27 and 3.28):

\[
R_v = \mu \frac{d^*}{2 \delta a} = \frac{1}{2 \beta v} \sqrt{\frac{t_0}{\pi}} \quad C_v = \frac{\xi}{p_{\text{sat}}} d^* = b_v \sqrt{\frac{t_0}{\pi}} \tag{4.22}
\]

and

\[
\frac{1}{\omega C_v} = \frac{t_0}{2\pi C_v} = R_v
\]

The moisture content in a material changes when the relative humidity (or capillary pressure) changes. The result of this is that not only changes of the water vapour pressure but also of the temperature can cause moisture to flow. To solve this problem at first the heat conduction equation has to be solved to get $p_{\text{sat}} = p_{\text{sat}}(x, t)$. What is left then is the highly non-linear vapour diffusion equation.

The different penetration depths for temperature and moisture content can cause tensions in the material by thermal and moisture expansion. The linear thermal expansion coefficient is defined as:

\[
\alpha = \frac{1}{\ell} \frac{d\ell}{dT} \tag{4.23}
\]

where: $\ell = \text{length}$

The moisture expansion coefficient is defined analogously:
$\alpha_v = \frac{1}{\ell} \frac{d\ell}{d\varphi}$

(4.24)

$\alpha_v$ is strongly dependant on the relative humidity.

Suppose an exterior wall is insulated at the exterior side with a vapourtight material and finished with a thin porous plaster. The wall is south oriented and the temperature has a large daily variation by the solar radiation absorption. The layer is thin so the temperature difference across the plaster is small. Hence, if the exterior absolute humidity is as good as constant the relative humidity in the plaster will vary strongly. As the equilibrium moisture content is dependent on the relative humidity there will be a periodic moisture flow at the exterior surface. The hygric penetration depth is much smaller than or of the same order of the plaster thickness causing different moisture expansion at the exterior and interior surface of the plaster. The daily variation of the tension in the plaster by this daily moistening and drying process results in micro-cracking and often to damage after a long time. The damage risk can reduced with:

- small solar absorptivity
- drying at two sides, so no vapourtight insulation material or a cavity
- material that absorbs little moisture, has a small expansion coefficient and is sufficient elastic. Usually it is hard to predict whether damage will occur and empirical research is necessary.

In order to find an analytical solution the equation has to be linearised. Of course this is at cost of the accuracy. Two simplifications are introduced:

- The plaster is iso-thermal: everywhere the same temperature and the same as the surface temperature.
- The coefficients in the equations are constant.

The linearised equation with boundary condition is ($\varphi = p_e/p_{sat}(T_s)$):

$$\left( \frac{\xi}{p_{sat}} \right)_{mean} \frac{\partial \varphi}{\partial t} = \frac{\delta_a}{\mu} \frac{\partial^2 \varphi}{\partial x^2} \quad \text{with } \delta_a = 1.8 \cdot 10^{-10} \text{ s}$$

$$\beta \left( \frac{p_e}{p_{sat}(T_s)} - \varphi_s \right) = -\frac{\delta_a}{\mu} \left( \frac{\partial \varphi}{\partial x} \right)$$

Where: $\text{mean} = \text{mean value of the surface temperature } T_s$  
$p_e = \text{water vapour pressure of the air}$

Now the problem is completely the same as the thermal problem, but with instead of the air temperature a fictive relative humidity $p_e/p_{sat}(T_s)$.

Expansion wood at $\varphi = 0.5 \rightarrow 0.6$
Tangential 0.45 till 90%
Radial 0.30 till 60%

Example hygric tensions
An exterior wall has from exterior to interior the next assembly: concrete 10 cm, insulation material 10 cm, plaster 1 cm.

Data: Plaster: $\lambda = 0.3$ W/mK, $\rho = 15$ kg/m$^3$, $c = 840$ J/kgK, $\mu = 15$, $\xi = dw/d\varphi = 7$ kg/m$^3$, solar absorptivity $a = 0.8$

Insulation: $\lambda = 0$ and $\mu \to \infty$

On a sunny day the temperature outdoors is 8ºC (constant). The solar irradiance can be approximated by:

$$E_z = 300 \left(1 - \cos \frac{2\pi t}{t_0}\right) \text{W/m}^2 \text{ with } t_0 = 24 \cdot 3600 \text{ sec}.$$

Surface coefficient of heat transfer $h_e = 20$ W/m$^2$K. The relative humidity is constant and 80% ($\varphi = 0.8$). The surface coefficient of vapour transfer $\beta_e = 10^{-7}$ s/m.

a. Calculate the temperature of the plaster as a function of time.

$$T_{rev} = 8 + 300 \cdot 0.8 \left(1 - \cos \frac{2\pi t}{t_0}\right)/20 = 20 - 12 \exp(j\omega t)$$

RC-circuit: $R = 1/20 + 0.01/(2 \cdot 0.3) = 0.067$ m$^2$K/W, $C = 15 \cdot 840 \cdot 0.01 = 126$ J/m$^2$K, $1/\omega C = 24 \cdot 3600/(2\pi \cdot 126) = 109$ m$^2$K/W

$$T_s = 20 \cdot (0.017 - 109j) \cdot 12 \exp(j\omega t)/(0.067 - 109j) \approx 20 - 12 \exp(j\omega t)$$

Hence $T_s \approx T_{rev}$ (no gradient)

b. Calculate the maximum difference of the relative humidity between the front and the backside of the plaster.

The fraction $p_e/p_{sat}$ can be approximated by a sinus function:

$$(p_e/p_{sat})_{max} = 0.8, \quad (p_e/p_{sat})_{min} = 0.8 \cdot p_{sat}(8)/p_{sat}(32) = 0.8 \cdot 1074/4762 = 0.18$$

so $p_e/p_{sat} = 0.49 + 0.31 \cos(2\pi t/t_0)$

Moisture resistance: $Z = 10^7 + 0.01 \cdot 15/(2 \cdot 1.8 \cdot 10^{-10}) = (1 + 41.7) \cdot 10^7$

Mean moisture capacitance: $C_v = (7/1074 + 7/4762) \cdot 0.01/2 = 4 \cdot 10^{-5}$, $1/\omega C_v = 34.4 \cdot 10^7$ m$^2$K/W

The difference of the relative humidity across the moisture resistance in the material is: $\varphi = 0.49 + (41.7/(1 + 41.7 - 34.4j)) \cdot 0.31 \exp(j\omega t) \rightarrow$ the amplitude is: $|\Delta \varphi| = 0.67 \cdot 0.31 = 0.235$

The maximum difference is two times the amplitude so: 0.47.

c. Calculate the maximum variation of length per linear metre if the thermal expansion coefficient is $10^{-5}$ K$^{-1}$.

$$\Delta \ell = 2 \cdot 12 \cdot 10^{-5} = 0.24 \text{mm/m}$$

d. Calculate the maximum difference per linear metre between front and backside of the plaster if for hygric expansion is given:

$$\alpha_v = \frac{1}{\ell} \frac{\Delta \ell}{\Delta \varphi} = 2.7 \cdot 10^{-3} \quad \Delta \ell = 2.7 \cdot 0.47 \cdot 10^{-3} = 1.27 \text{mm/m}$$
4.5.2. Humidity in a room

In a room temperature fluctuations are smaller if thermal mass is present (section 3.3.4). Similarly by the presence of hygroscopic materials the relative humidity is more constant.

Suppose that the air in the room is fully mixed. The moisture balance of the air yields the next equation:

\[
G = \frac{\dot{V}}{R_vT} \left( p_i - p_g \right) + \frac{\dot{V}}{R_vT} \frac{dp_i}{dt} + \sum_k A_k g_k
\]  \hspace{1cm} (4.25)

where:
- \( \dot{V} \) = ventilation rate (m³/s)
- \( G \) = vapour production (kg/s)
- \( V \) = air volume (m³)
- \( g_k \) = density of vapour flow rate to surface \( k \).

By the complexity of the transient vapour transport in hygroscopic materials the equation can only be solved by the computer. In order to understand certain transient effects the density of vapour flow rate \( g_k \) is estimated by the expression:

\[
g_k \approx C \frac{p_i}{dt} - \frac{\xi d^*}{p_{sat}} \frac{dp_i}{dt}
\]

Hence the vapour flow through the wall is neglected just as the influence of the surface coefficient of vapour transfer and the vapour resistance of the wall. Because of that the equation is simplified to a first order differential equation that can be solved easily if the product \( \xi d^* \) is known. A possibility is to estimate \( d^* \) with the penetration depth for 24 hour variations. As this equation is mainly used for step changes in vapour production and one is interested then in the change the first 6 hours, the \( d^* \) is usually based on a smaller period (e.g. 6 hours).

Apart from the walls also moisture is also stored in furniture and furnishings. For the soft furnishings a value from literature is \( \xi d^* \approx 0.028 \text{ kg/m}^2 \) and for furniture \( \xi d^* \approx 0.023 \text{ kg/m}^2 \).

The equation has for a constant \( p_e \) and a stepwise change of vapour production the solution (see section 3.3.4):

\[
p_i(t) = p_{i0} + \left( p_{i0} - p_{i\infty} \right) \exp \left(- \frac{t}{\tau} \right)
\]  \hspace{1cm} (4.26)

where
- \( \tau = \frac{V}{V + R_vT \sum_k A_k \xi_d / p_{sat}} \)
- \( t = 0 \) \( p_{i0} = p_e \)
- \( t \to \infty \) \( p_{i\infty} = p_e + \frac{R_vT}{V} G \)

So the time constant \( \tau \) increases by the hygroscopic walls resulting in a slower increase of the relative humidity.
The influence of walls on the humidity can also be estimated for sinusoidal fluctuations. The equation then reads (see section 3.3.5, complex notation):

\[
\tilde{G} = \frac{V}{R_v T} (\tilde{p}_e - \tilde{p}_c) + j \omega \frac{V}{R_v T} \tilde{p}_i + \sum_k \frac{A_k}{Z_k} \tilde{p}_i \quad j^2 = -1
\]

with:

\[
\tilde{G} = G - \bar{G} \quad ; \quad \omega = \frac{2\pi}{t_0}
\]

\[
\text{So } \tilde{p}_i = \frac{\tilde{p}_e + \frac{R_v T \tilde{G}}{V}}{1 + j \frac{2\pi V}{t_0} + \frac{R_v T \sum A_k}{V} \sum_k \frac{1}{Z_k}} \quad \text{with } Z_k = \frac{1}{\beta_k} + \frac{1-j}{2b_v \sqrt{\pi}}
\]

**Example**

*Step response* start \( p_0 = p_e = 700 \text{ Pa}, n = 0.3 \text{ h}^{-1}, G = 400 \text{ gr/h}, V = 100 \text{ m}^3, T = 290 \text{ K}, R_v = 462 \text{ J/kgK}, \) walls: 140 m\(^2\) with \( \xi_d = 0.02 \) kg/m\(^2\), furniture: 10 m\(^2\) with \( \xi_d = 0.023 \) kg/m\(^2\)

\[
\tau = \frac{1}{0.3} + \frac{462 \cdot 290}{0.3 \cdot 100} \left( \frac{140 \cdot 0.02}{1938} + \frac{10 \cdot 0.023}{1938} \right) = 10.3 \text{ hours}
\]

\[
p_{10} = 700 + \frac{400 \cdot 10^{-3}}{3600} \frac{462 \cdot 290}{30 / 3600} = 2486 \text{ Pa}
\]

The saturation pressure at 17°C is 1938 Pa. The pressure \( p_e \) will never be reached, as that is higher than the saturation pressure. Suppose that at 80% R.H. there is condensation on cold surfaces. After how much time will this happen?

\[
t = \tau \ln \frac{p_{10} - p_0}{p_{10} - p_i} = \frac{2486 - 700}{2486 - 0.8 \cdot 1938} = 6.66 \text{ hours}
\]

*Same room but with a variation:* \( \tilde{p}_e = 0, \tilde{G} = 200 \text{ gr/h}, t_0 = 24 \text{ h} \)

All walls are equal: \( b_v = 12 \cdot 10^{-8} \text{ kg}^{1/2} \text{s}^{1/2} \text{ J}^{-1/2} \), \( \beta = 1.4 \cdot 10^{-8} \text{ s/m} \)

\[
\text{So } \sum \frac{A_k}{Z_k} = \frac{140 \cdot 10^{-8}}{7.6 - 6.9 j} \cdot \frac{7.6 + 6.9 j}{7.6 + 6.9 j} = 140 \cdot 10^{-11} (72 + 65.5 \text{ j})
\]

\[
\tilde{p}_i = \frac{462 \cdot 290}{30} \cdot 200 \cdot 10^{-3} e^{j \omega t}
\]

\[
\tilde{p}_i = \frac{893}{|2.62 + j 2.35|} = 254 \text{ Pa}.
\]

If there were no 'capacitances the fluctuation had been 893 Pa instead of 254 Pa. So the 'damping' is 0.28. Especially walls cause a reasonable constant indoor vapour pressure.
4.5.3. Surface humidity

Exposed to fluctuating relative humidity the mean relative humidity of a surface layer can be higher than that of the air. There are several explanations for:

a. The surface is gets periodically wet and has not sufficient time to dry.

b. The non-linearity of the equations. E.g. uptake of moisture is quicker than drying (see section 3.3.3).

To investigate if a surface layer gets wetter and wetter the problem can be reversed i.e. one investigates if a wet surface gets dryer.

Suppose that during time \( t' \) the mean density of vapour flow rate to the surface is \( g^+ \) and that the drying time is \( t' \) with flow rate \( g^- \). So the surface gets dryer if \( g^- t < g^+ t' \).

Or: \[ \beta (p_{\text{sat}}(T_{is})-p_{\text{sat}}(T_{s}))t < \beta (p_{\text{sat}}(T_{s})-p_i)t' \] (4.27)

It is assumed that the relative humidity of the surroundings is 100% during wetting and that the moisture resistance of the surface layer can be neglected. Especially for drying this is arguable (see section 4.3.3). Drying is slower so this assumption is optimistic. The surface temperature \( T_s \) and the surface coefficient for vapour transfer are supposed to be different for drying and wetting.

A situation where equation 4.27 can be applied is the wetting and drying of a hygroscopic ceiling (e.g. gypsum board) in a shower cabinet. During showering the air near the ceiling is saturated and has a higher temperature than the ceiling. Moreover this air contains many small droplets that have a similar effect as a very large surface coefficient. During the drying period the surface coefficient is much smaller and the temperature lower. So the vapour flow is much smaller. The problem is how can be prevented that moulds appear at the ceiling. The equation 4.27 shows that a high surface temperature is important. A good insulation of the ceiling and (radiative) heating also during the drying time might help. Suction of air during showering is often not sufficient as the air movement can still be very low near the ceiling (small ventilation efficiency) and around the grill much moisture is absorbed resulting in a problem. Ventilation during the drying time helps, as by ventilation the vapour pressure will be lower. In a big room (e.g. high ceiling) the density of vapour flow rate to the ceiling is smaller because there is a larger area for the same flow and the vapour pressure is less quick saturated. It is merely a problem of small cabinets. Avoidance of hygroscopic material is often the only solution.

Suppose the density of vapour flow rate varies sinusoidal around a mean value or \[ g = -\frac{\delta_aP_{\text{sat}}}{\mu} \frac{\delta \phi}{\delta x} = -\frac{\partial}{\partial x} \left( \int \frac{\delta_aP_{\text{sat}}}{\mu} d\phi \right). \] So in average the mass of moisture in the material doesn’t change. As the integral is a non-linear function of the relative humidity the mean of this property is higher by the variation; so also the accompanying moisture content) is by this higher than without variation (see figure). This is but one example of
the effect of non-linearity. In general a transient computer model is necessary to get accurate results.

A bathroom is used 2 hours a day: 1 hour in the morning and 1 hour in the afternoon \( (t^+ = 3600 \, s, \, t^- = 3600 \times 11 \, s) \). When the room is used the relative humidity is 100% and on all surfaces there is condensation, especially on the gypsum cold ceiling (non heated room above the ceiling).

The pressure difference during condensation is \( p_{\text{sat}}^+ - p_{\text{sat}}(T^+) = 840 \, \text{Pa} \). During the 11 hours the temperature decreases until \( 10^\circ \text{C} \) and the mean difference \( p_{\text{sat}}(T^-) - p_i = 125 \, \text{Pa} \) (R.H. = 90%). The surface coefficient for vapour transfer is supposed to be constant all day.

There is no accumulation of moisture as: \( 1 \cdot 840 < 11 \cdot 125 \). All the same in this humid bathroom with R.H. = 90% and a ceiling that is virtually wet all the time moulds will appear.

4.5.4. Interstitial condensation by vapour diffusion

The oldest method to investigate whether measures are necessary to prevent interstitial condensation is the dewpoint method. One calculates the course of the temperature in the structure with given boundary conditions and determines whether the dewpoint temperature of the indoor air is situated at the warm or the cold side of the insulation layer. If the dewpoint is at the warm side (so the temperature is almost everywhere lower than the dewpoint) a vapourretarder is needed.

Around 1950 Glaser developed his method for freezers. Different from building walls, the walls of freezers are completely airtight, consist of non-hygrosopic and not-capillary materials and there is steady state one-dimensional vapour transfer. So it is doubtful whether this method is effective for building structures.

If there is steady state vapour transport with constant boundary conditions, i.e. independent of the moisture content, one gets for the one-dimensional density of vapour flow rate in absence of condensation:

\[
g = \frac{p_i - p_e}{Z_{\text{tot}}} ; \quad Z_{\text{tot}} = \sum_{k=1}^{n} \frac{\mu_k d_k}{\delta_k} + \frac{1}{\beta_i} + \frac{1}{\beta_e}
\]

where \( g = \) density of vapour flow rate
\( p_i \) and \( p_e \) = vapour pressure of the air at both sides
\( Z_{\text{tot}} \) = total vapour resistance
\( n \) = number of layers

Similar to eq.2.5 the vapour pressure at the interface of layer \( k \) and layer \( k+1 \) (from outdoors to indoors) is:
\[ p_k = p_e + (p_1 - p_e) \left( Z_e + \frac{\sum_{k=1}^{k} Z_k}{Z_{tot}} \right) \]  

Similarly the temperature is:

\[ T_k = T_e + (T_1 - T_e) \left( R_e + \frac{\sum_{k=1}^{k} R_k}{R_{tot}} \right) \]  

If the temperature in the structure is below the dewpoint condensation will occur; so if \( p_{sat}(T_k) < p_k \). The interface where this difference is maximal is called the critical interface. Due to condensation the vapour pressure at the critical interface will be the saturated one: \( p_{sat}(T_c) \left( \leq p_k \right) \).

So all vapour pressures in the structure will be lower (see adjacent figure). If the vapour resistance between the critical interface and the outdoor air is \( Z_{ce} \), the difference between the density of vapour flow rate to the critical interface and from it equals:

\[ g = \frac{p_1 - p_{sat}(T_c)}{Z_{tot} - Z_{ce}} - \frac{p_{sat}(T_c) - p_e}{Z_{ce}} \]  

where: \( T_c = \) the temperature of the critical interface.

\( g = \) the net density of moisture flow rate due to condensation [kg/m³s]

In this way one can determine whether at given boundary conditions there is condensation and how much per m² per second. In reality there are alternately periods of condensation and drying. In order to determine whether or not a structure is ‘safe’ the lengths of these periods and the corresponding boundary conditions must be appointed. Besides a criterion needed, e.g. no condensation is permitted or the condensation must stay below a fixed threshold.

To date there is much criticism on the Glaser method and computer models that solve the coupled transient heat and moisture and air differential equations will replace this method. The criticism concerns:

- In structures with airflow the Glaser method gives wrong results.
- In a structure more than one wet plane is possible and drying and wetting can occur at the same time For example if initial moisture is trapped in a structure during construction (construction damp, precipitation) this moisture can migrate to exterior side of the insulation during winter and to the interior in summer (summer condensation). The Glaser method doesn’t reckon with initial moisture and there is only one condensation interface.
- If adjacent to the critical interface the materials are hygroscopic, the moisture content in these materials just before condensation is increased very much as at high relative humidity there will be much capillary condensation. The Glaser method neglects the moisture storage in materials. A correction is possible however, by choosing a lower pressure than the saturation pressure as the maximum e.g. 0.98p_{sat}. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{criticism.png}
\caption{Criticism on the Glaser method}
\end{figure}
- If the adjacent materials are capillary the condensed water at the critical interface will be sucked into these materials. So there is also water transport.
- The temperature distribution changes partly by the dependence of the thermal conductivity on moisture content and partly by latent heat of condensation or evaporation.

Often the objective is not a realistic description of the moisture distribution but a simple answer on the question: is the structure safe, given a certain safety criterion? The answer can be found in most cases with a simple model.

Analogous to the reasoning used for the steady state heat transport also here the transport can be considered steady state if the quantities are read as averages over a sufficient long time. Here one month is chosen and

- Capillary condensation and transport will be neglected
- The temperature is not influenced by moisture
- The critical interface is wet all the time (R.H. = 100%).

The last point is correct during condensation, but not all the time during drying. So in reality there is less drying. However if a structure is not safe with the maximum drying it will surely not be safe with less.

Not safe means there is an annual excess of moisture (resulting condensation) and the structure will get wetter and wetter in the course of years.

The monthly mean temperature of the critical interface can be calculated with:

$$\overline{T_c} = \left(1 \cdot \frac{R_{ce}}{R_{tot}}\right) \overline{T_{rev}} + \frac{R_{ce}}{R_{tot}} \overline{T_i}$$  \hspace{1cm} (4.32)

The monthly mean density of moisture flow rate to the wet critical interface is:

$$g = \frac{p_i - p_{sat}(T_c)}{Z_{tot} - Z_{ce}} - \frac{p_{sat}(T_c) - p_e}{Z_{ce}}$$  \hspace{1cm} (4.33)

The amount that condenses (positive value) or dries (negative value) per month is found by multiplying g with the time. A severe performance requirement is that there is no resulting condensation in either month. In that case it is sufficient to check whether or not there is resulting condensation in the worst month (January). Another criterion is that there is no resulting condensation in a year and that the maximum condensed water is below a threshold value. The annual resulting condensation is found by adding all monthly amounts. The sum of the positive values is the maximum condensed water.
How much condensed water is allowable depends on the materials near the critical interface. If these materials are capillary one can assume that the moisture is sucked. From the maximum allowable moisture content of the material and the thickness of the layer the maximal allowable amount at the interface can be deduced.

A safe value for wood is often about 0.1 kg/m² and for stony materials about 0.5 kg/m². If there is condensation against glass, a foil or metal there is a danger of droplets origination. These droplets can fall or slide down and accumulate somewhere. So to be safe the threshold is low: 0.05 kg/m².

As the vapour saturation pressure is a non-linear function of temperature the mean saturation pressure is higher than the saturation pressure at the mean temperature of the critical interface; in other words the mean saturation pressure is the saturation pressure at a higher than the mean temperature. This higher temperature is called the *equivalent temperature of condensation* ($T^*$):

$$p_{\text{sat}}(T) = p_{\text{sat}}(T^*)$$  \hspace{1cm} (4.34)

In order to calculate this temperature the variations around the mean temperature must be known. As the temperature variation outdoors is much larger (e.g. by radiation) than indoors the latter variation is neglected. Another argument is that the critical layer often is situated at the cold side so close to outdoors. In a good approximation the correction is:

$$T^*_c = \bar{T}_c + D^2 (T_{\text{rev}}^* - \bar{T}_{\text{rev}})$$  \hspace{1cm} (4.35)

If the influence of heat storage in the wall is neglected the factor $D$ can be approximated with:

$$D = 1 - \frac{R_{\text{ce}}}{R}$$  \hspace{1cm} (4.36)

The correction on $T_c$ is smaller if there is much heat storage (smaller fluctuations in the structure). A possibility to account for this is using daily mean temperatures for the determination of the monthly $T^*_{\text{rev}}$ instead of hourly ones. This leads to a more pessimistic and therefore safer judgment.

In the table overleaf climate data of De Bilt of 1977 are given. For the determination of the *equivalent effective temperature* for condensation for a north façade next data were used: solar absorptivity $a = 0.4$; albedo $= 0.2$ (reflectivity ground); $h_e = 25$ W/m²K; emissivity $\varepsilon = 0.9$. For the horizontal plane $a = 0.9$. The irradiation of the vertical north façade contains a part of the ground-reflected solar radiation. Then the atmospheric long-wave radiation is only half of the amount on a horizontal plane. The correction for the equivalent temperature is given for hourly and daily means.
Climate classes are introduced to characterize rooms in a simple way regarding humidity. As the class is higher the more severe are the requirements put on the structure to be safe. The Dutch classes are:

- **Class I:** \( p_i - p_e < 50 \text{ Pa} \).
- **Class II:** \( 50 < p_i - p_e < 290 \text{ Pa} \).
- **Class III:** \( 290 < p_i - p_e < 400 \text{ Pa} \).
- **Class IV:** \( 400 < p_i - p_e \).

### Examples of thresholds

- **a:** Calculate the maximum value of \((p_i - p_e)\) for which there is no resulting condensation in a north façade in January.

Data for the north façade: \( R_{ce}/R = 0.14/3.14; Z_{ce}/Z = 1; \Theta_i = 21^\circ\text{C};\) solar absorptivity \(a = 0.4;\) emissivity \(e = 0.9;\) outdoor climate De Bilt (see table above). (Note that from \(Z_{ce}/Z = 1\) follows \(p_i - p_e = p_{sat}(T_i) - p_e\))

With the method given above one finds for a correction based on hourly means: 75Pa and for a daily: 71Pa. So not much difference. Here the atmospheric radiation \(\Delta E_{at}\) is taken half of the horizontal one. If this is neglected one finds for daily means: 114 Pa, so much higher value. This radiation appears to be very important here contrarily with the thermal problem. Unfortunately in a complicated urban situation the value can only be estimated.
b: Calculate the maximum value of \((p_i - p_e)\) for which there is no resulting condensation in the same north façade in a year.

The results are: 287 Pa (hourly), 264 Pa(daily), 348 Pa without atmospheric radiation.

Again we see the importance of the atmospheric radiation.

c: Calculate the maximum value of \((p_i - p_e)\) for which there is no resulting condensation in a flat roof in a year. The solar absorptivity of the roof is \(a = 0.9\). All the other data are identical with the wall data.

The results for this horizontal surface are: 746 Pa (hourly), 542 Pa (daily), 906 Pa (without atmospheric radiation).

Remarkably is the difference here for hourly and daily means. The reason is the large fluctuation of the effective temperature due to the solar radiation. Also here the long wave atmospheric radiation is very important.

The extension of the Glaser method with condensation and/or drying at more than one place in a layered structure is difficult and will not be treated here.

Condensation doesn’t need to be restricted to interfaces but can happen also in a whole layer. The reason is the non-linear relation between saturation pressure and temperature. E.g. if at either side of a layer there is saturation but a different temperature there will be a saturated vapour pressure in the layer that follows the temperature. So if the temperature is a linear function of the place the vapour pressure is non-linear, i.e. the gradient at one side of the construction differs from the value at the other side. The result is that there must be moisture accumulation in the structure:

\[
\frac{\partial w}{\partial t} = \frac{\partial^2 p_{sat}}{\partial x^2} \quad \text{(4.37)}
\]

If the temperature course is linear the total moisture accumulation per m² per second is:

\[
g = \frac{\delta x}{\mu} \left[ \left( \frac{dp_{sat}}{dx} \right)_1 - \left( \frac{dp_{sat}}{dx} \right)_2 \right] = \frac{\delta x}{\mu} \frac{dT}{dx} \left[ \left( \frac{dp_{sat}}{dT} \right)_1 - \left( \frac{dp_{sat}}{dT} \right)_2 \right] \quad \text{(4.38)}
\]

Or:

\[
\frac{dp_{sat}}{dT} = \frac{L}{R_y T^2} p_{sat} \rightarrow g \approx -\frac{0.13 \cdot 10^{-10}}{\mu} \frac{dT}{dx} \left(p_{sat}(T_1) - p_{sat}(T_2)\right)
\]

Note that \(g\) is always positive.

This situation occurs in practice when moisture is locked up between two vapour tight layers. Dependent on the temperature distribution
moisture will migrate from one side to the other leaving every time a part behind in the layer. By this mechanism not only non capillary materials like mineral wool can get wetted but foams as extruded polystyrene ($\mu = 80$) as well.

During the construction of a roof consisting from inside to outside of concrete, an insulation layer and a watertight top layer (warm roof) it was raining and the roof was not covered. One didn’t take the time to let it dry afterwards and closed the still wet roof. So water was locked in. Normally one waits. This doesn’t mean that there is no moisture at all as there can still be some construction moisture from the concrete.

According Glaser the structure is safe, but is that really true?

If it is cold moisture will evaporate at the warm side of the insulation and condensate at the cold side. That means at the same time both sides are wet and moisture will accumulate slowly in the insulation. If it is warm and the sun is shining (surface temperatures of $50^\circ$C are not unusual) the moisture at the top of the insulation will evaporate and condensate below. So again there is accumulation in the insulation. Result more energy loss by the wet insulation, top layer can get loose etc. At the very long run one can expect that moisture will disappear by diffusion (large $\mu$-value) through the top layer. To accelerate this process one often makes ventilation holes in the roof. This is also risky (see section 4.5.5). It better to prevent locked up moisture, so covering the roof when it is raining, waiting long enough before closing the roof, using a vapour tight foil between concrete and insulation to prevent construction moisture entering the insulation.

Examples from practice:

- warm roof
  - During the construction of a roof consisting from inside to outside of concrete, an insulation layer and a watertight top layer (warm roof) it was raining and the roof was not covered. One didn’t take the time to let it dry afterwards and closed the still wet roof. So water was locked in. Normally one waits. This doesn’t mean that there is no moisture at all as there can still be some construction moisture from the concrete.

According Glaser the structure is safe, but is that really true?

- summer condensation
  - If it is cold moisture will evaporate at the warm side of the insulation and condensate at the cold side. That means at the same time both sides are wet and moisture will accumulate slowly in the insulation. If it is warm and the sun is shining (surface temperatures of $50^\circ$C are not unusual) the moisture at the top of the insulation will evaporate and condensate below. So again there is accumulation in the insulation. Result more energy loss by the wet insulation, top layer can get loose etc. At the very long run one can expect that moisture will disappear by diffusion (large $\mu$-value) through the top layer. To accelerate this process one often makes ventilation holes in the roof. This is also risky (see section 4.5.5). It better to prevent locked up moisture, so covering the roof when it is raining, waiting long enough before closing the roof, using a vapour tight foil between concrete and insulation to prevent construction moisture entering the insulation.
An inverted roof has the watertight layer below the insulation, so water can come below the insulation. This means the insulation material must withstand influences of the outdoor climate and can surely not be of a capillary material (so e.g. closed cells).

A situation is possible that during a long time there is water above and below the insulation e.g. by a too small slope and too little drying. By the temperature difference across the insulation moisture will penetrate and accumulate. This is a very slow process especially in closed cells materials like XPS.

Important is the time the upper part is wet. Except from the slope is also dependant on ballast layer as tiles gravel. With gravel the drying is better so less problems. Even without problems the effective thermal resistance of the insulation material in such roof is 10 till 20% smaller than the value of the dry material.
Questions

Mention some physical processes that can cause rain penetration through brickwork. Which process is probably the most important? Mention some causes of rain penetration through brickwork. Describe what happen.

How do you explain the very different moisture content in a mortar joint and the adjacent bricks even when there is no moisture transport? Why is there no temperature jump at the same interfaces?

A wall consists of, from exterior to interior: concrete, mineral wool, and gypsum board. On a cold day condensation starts at the interface of concrete and insulation.

a. Why is there condensation and how can it be prevented?

b. What happens with the moisture?
   - The moisture remains on the concrete
   - The moisture drips down?
   - The moisture is sucked into the concrete
   - The moisture is sucked into the mineral wool.

Comment each assertion shortly: is it possible and if yes are there conditions or is it impossible and why? Which combination of two or more of the assertions are most probable and why? (magnitude of the moisture flow rate, material properties). Is interstitial condensation always a problem? Which of the above mentioned options leads to the greatest problem?

An inverted roof has water below and above the insulation. The sun is shining and the water above the insulation starts to evaporate. There is also condensation everywhere in the insulation layer. Explain this phenomenon.

What is the influence of hygroscopic materials on the indoor climate? How thick is the surface layer that is responsible?

What are the maximal, the capillary, the critical and the hygroscopic moisture content? Which property (-ies) is difficult to determine experimentally and how is it done?

If one starts to dry a wet brick does it make a difference for a simple calculation if is a very quick drying or a slow one? What is assumed in the different cases for the model?

Which material has the largest height for rising dampness: a material with many big pores or a material with many small pores? Why do you expect that?

How can the water sorption coefficient be determined?
Which situation leads to the shortest drying time of a wet brick: if it is dried with dry air blown along the stone or if the stone is heated up to 100°C? Explain.

What is the meaning of the critical moisture content in relation to the open porosity for the drying behaviour of materials?

A flat roof consisting of 15cm concrete is insulated at the topside with 10cm extruded polystyrene. It is a so-called inverted roof, i.e. below the insulation there is a bituminous layer and above a ballast layer. Water is drained above and below the insulation.

\[ \lambda_{\text{concrete}} = 1.5 \text{ W/mK} \quad \lambda_{\text{is}} = 0.04 \text{ W/mK} \]
\[ h_i = 7 \text{ W/m}^2\text{K}, \quad h_e = 25 \text{ W/m}^2\text{K} \]

a. Calculate the thermal transmittance (U-value) of the roof (effect of ballast and the drains can be neglected)

During a shower of 15mm/hour water comes below the insulation causing extra heat loss. Suppose the rain is totally drained below and the temperature of the rain is the same as the outdoor air temperature.

b. How much is the extra heat loss if the outdoor air temperature is 5°C, indoor 20°C and the temperature of the water in the drain 12°C ($c_w = 4187 \text{ J/kgK}$)

c. What is now the apparent U-value?

After the shower there is a layer of water above and below the insulation. Suppose the temperature of these layers is the same as the temperature that should have been there without rain (outside 5°C, inside 20°C)

d. Calculate the density of vapour flow rate in the insulation at the bottom and at the topside.

\[ p_{\text{sat}} = \exp (65.8094 - 7066.27/T - 5.976 \ln T) \; ; \mu = 164 \; ; \delta_A = 1.8 \cdot 10^{10} \text{ s.} \]

What do you expect to happen? Hint: \[ \frac{dp_{\text{sat}}}{dx} = \frac{d}{dT} \frac{dp_{\text{sat}}}{dT} \]

Everything is the same as above except there is sunshine now. The irradiance is 500 W/m², the solar absorptivity 0.9

e. Calculate the effective temperature and the density of vapour flow rate in the insulation at the bottom and at the topside. What happened by the solar irradiance?

f. What is the influence of water at the top and bottom on the heat transport and why?

g. What are the benefits of an inverted roof and how can the problems like the ones below d. and e. be prevented.
5 Convective Heat and Moisture Transfer

5.1 Introduction

Air can flow in and through building envelope parts. The heat and moisture transport as a result of this flowing air is often more important than the transport by radiation, heat conduction and diffusion.

The air flow through the envelope can contribute significantly to the ventilation of a room (the so-called infiltration). Furthermore airflows are in rooms important for the thermal comfort of people.

The airflow can be intentional (purpose provided), e.g. to remove moisture in wall cavities or heat in glazing cavities, or adventitious e.g. infiltration through joints and cracks.

Airflows are caused by pressure differences. A classification of flows is based on the origin of these pressures: mechanical i.e. by ventilators or natural by wind or gravity (buoyancy). Buoyancy is caused by the gravity forces on volumes with different fluid densities. In most cases density differences are due to temperature gradients. The pressures arising from temperature differences are the so-called stack pressures. Another division is forced convection (wind, ventilators) and free convection (caused by buoyancy forces).

The prediction of air flows is very difficult. The accuracy of the solution of the differential equations for heat, momentum and mass flow (Navier Stokes equations) is often very limited by the calculation power of computers. Moreover the boundary conditions for the equations are very inaccurate, e.g. the dynamic pressures by wind, the influence of mechanical ventilation systems, data for the geometry of openings and cracks.

There is much literature with empirically determined formulas to estimate the heat and moisture transport by convection. To date it is possible to improve the accuracy of convective transfer by computational fluid dynamics (CFD), but much research has still to be done.

The objective in this chapter is to get some understanding of the mechanisms which cause flows and of heat and moisture transfer by convection. Quantitative results will be obtained by simplifications. The great number of empirical formulas that can be found in literature e.g. for surface coefficients, will not be listed.
5.2. Conservation of mass and of momentum

In chapter 1 the conservation of mass was treated. If there are no mass sources e.g. no chemical reactions the equation is:

\[ \dot{G}_{\text{out}} + \frac{dm}{dt} = G_{\text{in}} \]  
(5.1)

Applied to a differential control volume equation 1.5 with \( S_{m} = 0 \) can be derived. After substitution of \( w = \rho \) (the density of the fluid) and \( \vec{G} = \rho \vec{v} \) (the density of mass flow rate with \( \vec{v} \) the velocity vector of the fluid) the equation reads:

\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{v}) = 0 \]  
(5.2)

This is called the continuity equation.

E.g. Mass conservation requires that the steady state mass flow rate keeps the same through any cross-section of a tube or:

\[ G_{1} = \rho_{1} \dot{V}_{1} = \rho_{2} \dot{V}_{2} \]  
(5.3)

where: 1 and 2 denote two different places along the tube and \( \dot{V} \) the (volume) flow rate.

The (volume) flow rate is:

\[ \dot{V} = A_{\perp} \vec{v} \]  
(5.4)

The plane with surface area \( A_{\perp} \) is perpendicular to the flow direction. \( \vec{v} \) is the surface averaged flow velocity of the cross-section.

From mass conservation follows that in a tube the highest mean velocity will be found at the place with the smallest cross-section.

In the same way as heat flow lines were introduced also stream lines can be defined. In any place in the fluid the direction of the flow at a certain time is the tangent of a streamline. In a steady state the streamline is the path of a small volume of the fluid. If the flow is changing in time the path (different moments in time) is not the same as a streamline!

So in a steady state there is no velocity component perpendicular to streamlines. This means that the mass flow rate in a tube formed by streamlines is the same everywhere in this tube. So at places where streamlines are closer the velocity is higher. This is completely analogous to the heat flow density between heat flow lines.

From Newton’s laws of motion the conservation of momentum can be derived. The conservation of momentum for a flow states that the sum of all forces acting on a control volume is equal to the net momentum:
flow rate out of the volume (out-in) plus the change per unit of time of
the momentum of the fluid in the volume. For a small volume:

\[ G_{\text{out}} \vec{v}_{\text{out}} - G_{\text{in}} \vec{v}_{\text{in}} + \frac{d(m\vec{v})}{dt} = \sum_{i} \vec{F}_i \]  

(5.5)

The net momentum flow \( (G_{\text{out}} \vec{v}_{\text{out}} - G_{\text{in}} \vec{v}_{\text{in}}) \) rate represents the inertia
force of the flow. Note that if \( G=0 \) and \( m=\text{constant} \) this equation is the
same as Newton’s second law: \( \vec{F} = m\vec{a} \) (\( \vec{a} = \text{acceleration} \)).

The forces on the control volume can be divided into surface forces and
volume forces. In building physics the surface forces are caused by the
static pressure \( p \) and the friction (normal and shear stresses due to
gradients in the flow of a viscous fluid). The main volume force is
caused by gravitation:

\[ \vec{F}_g = -m g \vec{e}_z \]

\( g = 9.81 \text{ m}^2/\text{s} \) (acceleration due to gravity)

The momentum equation for a point in the fluid is obtained by dividing
the balance equation for a differential volume by that volume (similar to
the way the continuity and the heat conduction equation were obtained).
For example:

\[ \lim_{\Delta V \to 0} (G_{\text{in}} - G_{\text{out}}) \vec{v}_x / \Delta V = -\text{div}(\rho \vec{v}_x) \]

Using the continuity equation the derived momentum equation can be
written as:

\[ \rho \frac{\partial \vec{v}}{\partial t} + \rho v_x \frac{\partial \vec{v}}{\partial x} + \rho v_y \frac{\partial \vec{v}}{\partial y} + \rho v_z \frac{\partial \vec{v}}{\partial z} = -\text{grad} p + \text{div}(\vec{\tau}) - \rho g \vec{e}_z \]

where:

\[ \text{div}\vec{\tau} = \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) + \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) + \left( \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) \]

For Newtonian fluids:

\[ \tau_{\alpha \beta} = \eta \left( \frac{\partial v_\alpha}{\partial \beta} + \frac{\partial v_\beta}{\partial \alpha} \right), \quad (\alpha, \beta) = (x, y) \text{ or } (x, z) \text{ or } (y, z) \]

\[ \tau_{\text{ext}} = 2\eta \frac{\partial v_\alpha}{\partial \alpha} - \frac{2}{3} \eta \text{div}\vec{v}, \quad (\alpha) = (x) \text{ or } (y) \text{ or } (z) \]
5.3 Flow through pipes, pores and cracks

5.3.1 Bernoulli

If s is a coordinate along a stream line then the equation for the motion on this line is:

\[ \frac{\partial p v_s}{\partial t} + \frac{\partial}{\partial s} \rho v_s^2 / 2 = -\frac{\partial}{\partial s} (p + g \int_0^b \rho dz) + (div \vec{v})_s \]  

(5.6)

where: \( (div \vec{v})_s \) represents the friction force in the direction of the flow

At two different places on this line:

\[ \rho_0 v_{s1}^2 / 2 + p_1 + g \int_0^{h_1} \rho dz = \rho_0 v_{s2}^2 / 2 + p_2 + g \int_0^{h_2} \rho dz + \Delta p_{fr} + \int_{s_1}^{s_2} \frac{\partial p v_s}{\partial t} ds \]

where: \( p_1 \) = static pressure at \( z = h_1 \), \( p_2 \) = static pressure at \( z = h_2 \) \( \Delta p_{fr} \) = pressure loss by friction between 1 and 2

This equation can be understood as an energy conservation equation: the kinetic energy \( 0.5 \rho v^2 \) + the potential energy \( (p + \rho gh) \) at one place is equal to that on another place downstream plus the friction loss (converts to heat) and the work due to acceleration. We will only consider steady state flows so the last term at the r.h.s. will be left out.

In building physics the flow velocity is sufficient low to assume the fluid to be incompressible. This doesn’t mean that the density is a constant. If a gas is heated the gas will expand so the density will decrease. With the ideal gas law and constant pressure one can see that the density is inversely proportional to the absolute temperature. So with \( \rho = \rho_0 T_0 / T \):

\[ \int_0^b \rho dz = \rho_0 \int_0^T \frac{T_0}{T} dz \]

\[ \rho v_{s1}^2 / 2 + p_1 + \rho_0 \int_0^{T_0} \frac{T_0}{T} dz = \rho v_{s2}^2 / 2 + p_2 + \rho_0 \int_0^{T_2} \frac{T_0}{T} dz + \Delta p_{fr} \]

(5.7)

\[ \rho \approx \rho_0 = \rho(z=0) \text{ and } T_0 = T(z=0) \]

The pressure caused by a temperature difference is called the stack pressure (the phenomenon is called stack effect). The air movement caused by density differences is called buoyancy or free convection.

For an isothermal flow, without friction (ideal fluid) eq. 5.7 changes into the well-known Bernoulli equation:

\[ \rho v_{s1}^2 / 2 + p_1 + g \rho h_1 = \rho v_{s2}^2 / 2 + p_2 + g \rho h_2 = \text{const} \tan \theta \]

(5.8)
Through a hole in the bottom of a (cylindrical) bucket water flows due to gravity to the environment. The cross-section of the hole is 1cm² and the cross-section of the bucket is 900cm². At t=0sec the water level is 40cm above the hole.

Derive the equation for the mean water velocity in the hole as a function of the water level. (The viscosity of the water can be neglected)

After how much time will the bucket be empty?

a. Bernoulli

$$\rho v_{s1}^2 + p_1 + gh_1 = \rho v_{s2}^2/2 + p_2 \quad v_{s1} = v_{s2}/900 \text{ and } h_2 = 0$$

So with \( p_1 = p_2 = p_{\text{environment}} \)

$$v_{s1} \approx \sqrt{2gh_1}$$

b. The mass change per second is:

$$\frac{d(h_{10} \cdot 900 \cdot 10^{-4})}{dt} = \rho v_{s2} \cdot 10^{-4}$$

So the equation is:

$$\frac{900 \, dh_1}{dt} = -\sqrt{2gh_1} \rightarrow h_1 = \left( \sqrt{h_{10}} - \sqrt{\frac{2g}{1800}} t \right)^2$$

After substitution of \( h_{10} = 0.4 \text{m}, h_1 = 0 \text{m} \) and \( g=9.8 \text{ m/s}^2 \)

\( t=257\text{sec} \quad (\text{With viscosity it would last longer}) \)

If a streamline ends on a solid body \( v_{s2}=0 \) and if friction and gravity can be neglected then:

$$\rho v_{s1}^2/2 + p_1 = p_2$$

A point on a surface where a streamline ends is called a stagnation point. So at a stagnation point the pressure is 0.5\( \rho v^2 \) higher than the pressure at some distance of the surface. 0.5\( \rho v^2 \) is called by the (dynamic) stagnation pressure.

At an exterior building surface the wind will increase the pressure at the windward side and decrease the pressure at the leeward side. The pressure at a point on the building envelope minus the static pressure far from the building (undisturbed) is calculated with the expression below

$$p - p_0 = C_p \rho v_{\text{ref}}^2/2$$

(5.9)
where: \( v_{\text{ref}} \) = reference wind speed (e.g. at ridge height)  
\( C_p \) = wind pressure coefficient  
\( P_0 \) = static pressure

Note that the pressure coefficient \( C_p \) can also be negative (leeward side). Assumed is that the pressure coefficient is independent of the wind speed i.e. the wind pressure distribution on the building envelope can be scaled with a stagnation pressure at some point in the undisturbed flow field. This is only a good approximation for very turbulent flows (so high velocities). For wind loading (only high velocities are interesting) the coefficients should differ from the ones for ventilation (all velocities) calculations. This is not done in practice.

The value of \( C_p \) depends on the wind direction and on the geometry of the building. Tables of \( C_p \) are given by the AIVC (Air Infiltration and Ventilation Centre). It is impossible however to predict wind pressures accurately for all wind directions and velocities on the complex geometry of a building surrounded by other buildings, also not with a CFD model.

To be able to estimate for each hour of a year the pressures the relation between the wind speed from a meteorological station nearby and the reference wind speed must be known. This will always be quite inaccurate by the influence of surrounding buildings and the terrain upwind. One often assumes a power law windprofile. This is physically not correct. The correct profile is logarithmic, but in fact this is only true for a large terrain upwind with the same roughness and above a certain height: \( 20z_0 + d \) (\( z_0 \) is the roughness length and \( d \) the displacement). For a town with low-rise buildings this height is about 30m. Below that height there is no good method.

The standardized wind data are the velocities at 10m height above a large terrain without obstructions with 0.03m roughness length (grass). The data are often from airports. In the Netherlands these sites are hard to find, so measured data are corrected. The corrected data are called potential wind speeds \( (v_{\text{pot}}) \). The correction procedure is as follows: the measured data are transformed to the speed at 60m high (meso velocity) with the real roughness of the terrain assuming a logarithmic profile. The 60m data (meso-scale) are transformed to 10m with \( z_0 = 0.03m \).

The logarithmic profile is:  
\[
\frac{u_*}{0.4} \ln\left(\frac{z - d}{z_0}\right)
\]

So the velocity at the height \( h \) above the meteostation is:

\[
v(h) = \frac{\ln\left(\frac{60}{0.03}\right)}{\ln\left(\frac{10}{0.03}\right)} v_{\text{pot}} \frac{\ln\left(\frac{h}{z_{\text{meteo}}}\right)}{\ln\left(\frac{60}{z_{\text{meteo}}}\right)}
\]

So e.g. if \( z_{\text{meteo}}=0.03m \) (the standard \( z_0 \)) \( v_{\text{pot}} = v_{\text{measured}} \) and if \( v_{\text{pot}} = 10m/s \) \( v(6)=9.1m/s \). If the real \( z_{\text{meteo}}=0.1m \) then at the meteostation the
velocity at 6m is: \( v(6) = 8.4 \text{ m/s} \). If one assumes that the mesovelocity in town is the same as at the station (not true) then with \( z_{\text{town}} = 0.5 \text{m} \) the velocity at 6m is: \( v(6) = 6.8 \text{ m/s} \).

There are methods to relate the mesovelocity at the meteostation to a mesovelocity at another location (e.g. Dutch directive NPR6097).

With a Pitot tube (a measuring instrument consisting of a right-angled tube, see figure) the stagnation pressure of a flow is measured and by this the velocity of the flow.

If the static pressure in a room at \( z = z_0 \) is equal to the static pressure outside at the same height the height is called the neutral plane. If we assume a small air velocity and a uniform temperature in the room then the pressure difference across the wall due to the stack effect is:

\[
p_{01} = p_1 + \rho_0 g \frac{T_0}{T_1} (z - z_0), \quad p_{02} = p_2 + \rho_0 g \frac{T_0}{T_2} (z - z_0)
\]

Or

\[
p_1 - p_2 = 9.81 \cdot 1.29 \cdot 273(\frac{1}{T_2} - \frac{1}{T_1})(z - z_0) \approx 3455(\frac{1}{T_2} - \frac{1}{T_1})(z - z_0)
\]

So above the neutral plane \( p_1 > p_2 \) if \( T_1 > T_2 \). Below this plane it is just the opposite.

The stack effect can be the most important cause for air infiltration in a building. By the stack effect also bad odors can spread throughout a building. It can also be positively used to get rid of heat in buildings and...
glass houses. An example is the solar chimney. In constructions where openings are designed entirely on the stack effect the wind can change the flow rate very much and even the flow direction can be reversed. There are special cases with a stack effect that outreaches the wind pressures as is shown by the example below

A high-rise building of 200m high has an atrium of the same height. The neutral plane is assumed to be at z=0 i.e. the atrium has a very open connection with the ground level. If the atrium temperature is 20 °C and the outside temperature is -20°C (Canada) how much is the pressure on the roof of the atrium?

\[ p_1 - p_2 = 1.2 \times 9.8 \times (20+20) \times 200/273 = 345 \text{ Pa} \]

This is a very high load. It is of the same order of magnitude as the stagnation pressure of a wind velocity of 25m/s (Beaufort 10 or storm). The roof has to be fixed very strong unless it will be lifted like a hot air balloon.

In a tube the velocity can be measured with tube section containing a constricted throat: a venturi. If the cross section of the tube is \( A_1 \) and of the venturi \( A_2 \) then:

\[ G = A_1 \rho v_1 = A_2 \rho v_2 \]  

(5.12)

So without friction:

\[ \rho v_1^2 / 2 + p_1 = \rho \left( \frac{A_1}{A_2} \right)^2 v_1^2 / 2 + p_2 \]

(5.13)

Or:

\[ (\frac{A_1}{A_2})^2 - 1) \rho v_1^2 / 2 = p_1 - p_2 \]

So with the measurement of the differential pressure the fluid velocity can be determined. Downstream of the constriction the pressure increases again till the value \( p_1 \) is reached if there were no friction. In reality there will be friction (see eq. 5.25).

Instead of the velocity upstream of the constriction \( v_1 \) also the velocity downstream of it \( v_2 \) can be used. For a Venturi measurement section it is important that it is smooth and the pressure loss by friction as small as possible.

The fluid exercises a force on the Venturi:

\[ G(v_2 - v_1) = A_1(p - p_\infty) - A_2(p_2 - p_\infty) + F \]  

(5.15)
F = force on fluid == -F on tube

With eq. 5.13 and some algebra the force on the tube is:

\[
F = (A_1 - A_2)(p_1 - p_\infty) - \rho v_i^2 \frac{(A_1 - A_2)^2}{2A_2}
\]  
(5.16)

Without friction the force in the second part of Venturi section will be the same but with opposite sign (the total force is zero). In reality the absolute value of the force before the constriction will be larger than behind.

The theoretical maximum efficiency of a wind turbine can easily be derived. In the ideal case no energy is lost by turbulence induced by the turbine. The effect of the rotating blades is that the straight parallel streamlines of the undisturbed wind diverge. So the tube formed by the streamlines and touching the perimeter of the plane of the rotating blades has a conical shape. With the conservation of mass the mass flow in this tube is:

\[
G = A_1 \rho u_1 = A_{\text{turb}} \rho u_{\text{turb}} = A_2 \rho u_2
\]

Where \( A_1 < A_{\text{turb}} < A_2 \) are resp. surface areas in front, at the turbine, and behind the turbine (see figure)

\( u_1 > u_{\text{turb}} > u_2 \) are resp. undisturbed wind velocity, velocity at the turbine, and behind the turbine (see figure)

The wind power of the area spanned by the blades is:

\[
\Phi_w = \text{force} \times \text{velocity} = (A_{\text{turb}} \rho u_1^2/2)xu_1
\]

The efficiency \( \eta \) of a wind turbine is defined as

\[
\eta = \Phi / \Phi_w
\]

The efficiency has a theoretical maximum and cannot be 100%. This will be shown below. The power transferred to the turbine is:

\[
\Phi = A_1 \rho u_1^3/2 - A_2 \rho u_2^3/2 = G(u_1^2 - u_2^2)/2
\]

The power extracted by the turbine is also equal to the product of force and velocity at the turbine. The net force on turbine (difference of momentum) is:

\[
F = G(u_1 - u_2) \text{ or } F = G(u_1 - u_2)u_{\text{turb}}
\]

So: \( \Phi = G(u_1 - u_2)u_{\text{turb}} = G(u_1^2 - u_2^2)/2 \)

Or: \( u_{\text{turb}} = (u_1 + u_2)/2 \) and \( G = A_{\text{turb}} \rho (u_1 + u_2)/2 \)

So the efficiency is:

\[
\eta = (1 - \gamma^2)(1 + \gamma)/2 \text{ with } \gamma = u_2/u_1
\]

The maximum efficiency is found by differentiation

\[
d\eta/d\gamma = 0 \rightarrow \gamma = 1/3 \rightarrow \eta = 16/27
\]

So the theoretical maximum efficiency of a wind turbine is 16/27
5.3.2 The friction pressure loss in a pipe

If the momentum conservation equation is applied to a steady state flow through a horizontal pipe with a circular cross-section it reads:

\[ 0 = \sum F_i - \pi R^2 (p_1 - p_2) + 2\pi R l \tau_{Re} \]

So:

\[ \frac{\Delta p_{Re}}{\ell} = \frac{p_1 - p_2}{\ell} = -2 \frac{\tau_{Re}}{R} \]

Where \( p \) = static pressure
\( \tau_{Re} \) = shear stress at the pipe surface
\( R \) = radius
\( l \) = length

The shear stress near the wall is proportional to the derivative of the velocity to the \( r \)-coordinate:

\[ \tau = \eta \left( \frac{\partial u}{\partial r} \right) \bigg|_{r=R} \]

The coefficient \( \eta \) is the viscosity of the fluid.

If the flow velocity is low the streamlines are smooth lines; the flow is 'layered'. Such a flow is called laminar.

For a laminar flow the velocity profile in the circular tube is parabolic:

\[ u = 2\bar{u}(1 - \frac{r^2}{R^2}) \]

By substitution of eq. 5.19 into 5.18 and 5.17 we find Poiseuille’s law:

\[ \bar{u} = \frac{1}{8} \frac{p_1 - p_2}{\eta \ell} R^2 \quad \text{or} \quad \frac{p_1 - p_2}{\ell} = \bar{u} \frac{8\eta}{R^2} \]

Note that by the same pressure difference per unit of length, the mass flow velocity in a tube is proportional to \( R^2 \). This explains why when water is absorbed by a porous material the large pores determine the place of the wetting front.

At a high flow velocity the flow particles don’t move in smooth lines anymore the lines are twisted. Also the velocity is fluctuating very rapid. Instead of the instantaneous velocity a mean value over an enough long time is more convenient. Stream lines drawn with this mean value are smoother but in fact are no real stream lines: there a fluctuating flow velocity perpendicular to these lines with a mean value 0. The (mean) velocity profile of a turbulent flow in a pipe is more flat and the shear stress at the wall is much higher than in the laminar case. The question is when is a flow turbulent? This will depend on the ratio of the inertia and the viscous forces. This ratio is expressed in a dimensionless number: the Reynolds number. From experiments is
shown that when the Reynolds number is larger than 2200-2300 the flow is turbulent.

The Reynolds number is defined as:

$$Re = \frac{\rho \bar{u}^2}{\eta / D} = \frac{\rho \bar{u}D}{\eta}$$  \hspace{1cm} (5.21)

where $D$ is a characteristic length

The relation between the pressure drop in a pipe and the velocity for the whole region (laminar $\rightarrow$ turbulent) is given by:

$$\Delta p_{fr} = \frac{\lambda}{2} \rho \frac{\bar{u}^2}{D}$$  \hspace{1cm} (5.22)

The friction factor $\lambda$ depends on the Reynolds number and the relative roughness. Numbers are given in the so-called Moody diagram.

For a laminar flow this factor follows from:

$$\lambda = \frac{64}{Re}$$  \hspace{1cm} (5.23)

Air that leaves a sharp edged opening of a tube has at a distance of about 20 diameters from the orifice a velocity of about zero. If the static pressure at that place and in the opening is about the same the friction loss at the opening is about:

$$\Delta p_{fr} = \frac{1}{2} \rho v_1^2$$  \hspace{1cm} (5.24)

With $v_1$ = the velocity in the opening.

More general:

$$\Delta p_{fr} = \xi \frac{1}{2} \rho v_1^2$$  \hspace{1cm} (5.25)
In order to decrease this friction loss the velocity at the opening should be lower. This can be done with a diffuser.

Equation 5.25 is also called the orifice flow equation. This equation is also used for the pressure drop in bends and branches and constrictions in pipes and ducts. The factor $\xi$ can be found in textbooks for many different configurations.

### 5.3.3 Friction pressure loss of apertures and cracks and gaps

Another way to write equation 5.25 is:

$$\dot{V} = AV = AC_d \sqrt{2\Delta p / \rho}$$

where: $C_d (=1/\sqrt{\xi}) =$ discharge coefficient 0.6…1

The air flow through cracks joints, gaps etc in the building envelope is usually calculated with the power law equation:

$$\dot{V} = C_{tot}\Delta p^n$$

(5.26)

where: $C_{tot} =$ flow coefficient (eq 5.25: $AC_d\sqrt{2/\rho}$)  
$n =$ flow exponent $1/2 < n < 1$  (eq 5.25: $n=1/2$)  
$\Delta p =$ pressure difference across the construction

If the opening is in a good approximation 2D, e.g. window seams, the coefficient $C_{tot}$ is proportional to the length (in the third direction) l. So $C_{tot} = 1 \cdot C_l$. Sometimes $C_{tot}$ is written as $AC_A$ with A the area of the cross-section of the opening, e.g. for open windows and doors. $C_l$ and $C_A$ are called the specific air permeability coefficients. In ducts often the in- and outflow openings, the possible bends and branches determine the total pressure drop and not the length of the pipe.

One would expect a flow direction dependency of $C_{tot}$ and $n$ but these properties are seldom available or are neglected. There are many formulas to calculate $C$ and $n$ with the geometry of the opening. For purpose designed apertures they might be known but for cracks the geometry is seldom known. Empirical data, e.g. as can be found in the data-base of the AIVC (Air Infiltration and Ventilation Centre) are most often used.

The air flow model of a whole building consists of equations like the ones above that are coupled by the air mass balances of the rooms. The results are the room pressures $p_{oi}$ and the mass flow rates through the openings. Also mechanical ventilation rates are part of these equations, e.g. suction can result in underpressure.

The air flow calculation in a ‘leak’ building is not very accurate.

- Exterior wind pressures are not well known
- The relation between pressure difference and air flow for many leakages is not known. For openings with a large vertical
dimension as open doors, windows, staircases etc. it is not much more than a guess. If a door is always open between two rooms the alternative is to consider the two rooms as one zone (equal air temperature).

- Temperature stratification has a strong effect on the stack effect but is not known.

- Air exchange means a strong coupling of rooms for heat and moisture balance and has a very large effect on these quantities. Also here linking rooms to one zone is sometimes more accurate.

A convenient way to solve the network with MATLAB is by means of a variational principle. Let:

\[
H(p_1, p_2, \ldots) = \sum_k \frac{C_k}{n_k + 1} \left[ p_i + 3455h_k \left( \frac{1}{T_i} - \frac{1}{T_j} \right) - p_j \right]^{(l+n_k)} +
\]

\[
\sum_i \frac{C_i}{n_i + 1} \left[ p_i + 3455h_i \left( \frac{1}{T_i} - \frac{1}{T_e} \right) - C_p \cdot 0.5 \cdot \rho \cdot v^2 \right]^{(l+n_i)} +
\]

\[
+ \sum_{i,j} p_i (\dot{V}_{mi\rightarrow j} - \dot{V}_{nj\rightarrow i}) + \sum_i p_i (\dot{V}_{mi\rightarrow e} - \dot{V}_{me\rightarrow i})
\]

where:

- \( p_i \) is the room pressures transformed to reference height (see figure)
- \( k \) is the number of an opening between two zones: \( j \) and \( l \), so for each \( k \) \( i \) and \( j \) get a value.
- \( L \) is the number of an opening between a zone \( i \) and outdoors \( e \)
- \( \dot{V}_{mi\rightarrow j} \): volume flow rates (extract and supply) by mechanical ventilation from zone \( i \) to zone \( j \) and from zone \( i \) to zone \( e \) (outdoors)

The mass balance for each room follows from:

\[
\frac{\partial H}{\partial p_i} = 0
\]

One can easily show that for all \( p_i : H \geq 0 \) and \( \frac{\partial^2 H}{\partial p_i^2} \geq 0 \). So by minimizing the function \( H \) (MATLAB has a very quick tool for it) the pressures are found and then the calculation of the airflows is straightforward. Some very small corrections are applied to get a 100% correct mass balance.
5.4 The ventilated cavity

5.4.1 The air flow in the cavity

Suppose that a cavity wall between two rooms has an opening at the height \( h_1 \) in room 1 with temperature \( T_1 \) and an opening at height \( h_2 \) in room 2 with temperature \( T_2 \). In room 1 the static pressure at \( h = 0 \) is \( p_{01} \) and in room 2: \( p_{02} \).

In the absence of dynamic pressure the pressure \( p_{02} \) is equal to \( p_{01} \) plus the pressure differences by buoyancy and friction:

\[
p_{02} = p_{01} - \rho g \frac{T_1}{T_i} h_1 - \Delta p_{fr1} + \rho g \frac{T_0}{T_{cav}} (h_1 - h_2) - \Delta p_{fr2} + \rho g \frac{T_0}{T_2} h_2 \quad \text{or} \quad \Delta p_{fr1} + \Delta p_{fr2} = p_{02} - p_{01} - \rho g \frac{T_0}{T_i} h_1 - \rho g \frac{T_0}{T_{cav}} (h_1 - h_2) + \rho g \frac{T_0}{T_2} h_2 - p_{02}
\]

where \( \Delta p_{fr1} \) and \( \Delta p_{fr2} \) are the pressure drop by friction across the openings (positive if flow goes from room 1 to 2, else negative)

\( \Delta p_{fr1} \) = pressure drop by friction across the openings

\( \Delta p_{fr2} \) = pressure drop by friction across the openings

\( g \) = acceleration due to gravitation (9.8 m/s\(^2\))

\( T_{cav} \) = average cavity air temperature

The pressure drop by friction in the cavity can often be neglected \((\Delta p_{fr2} \ll \Delta p_{fr1})\). The pressure drop across the openings is e.g.

\[
| \Delta p_{fr1} + \Delta p_{fr2} | = \xi_{in} \frac{1}{2} \rho (\hat{V}/A_{in})^2 + \xi_{out} \frac{1}{2} \rho (\hat{V}/A_{out})^2
\]

where:

- \( \xi_{in} \) = inflow opening
- \( \xi_{out} \) = outflow opening

If the openings are both at the exterior:

\[
p_{01} - p_{02} = \frac{1}{2} (C_{p1} - C_{p2}) \rho v_{ref}^2
\]

and \( T_1 = T_2 = T_e \)

\[
\Delta p_{fr1} + \Delta p_{fr2} = \frac{1}{2} (C_{p1} - C_{p2}) \rho v_{ref}^2 + \rho g (\frac{T_0}{T_e} - \frac{T_0}{T_{cav}})(h_2 - h_1)
\]

with: \( v_{ref} \) = reference wind speed (for example, on ridge/roof height).

If the openings are both at the interior side:

\[
p_{01} - p_{02} = 0
\]

and \( T_1 = T_2 = T_i \)

\[
\Delta p_{fr1} + \Delta p_{fr2} = \rho g (\frac{T_0}{T_i} - \frac{T_0}{T_{cav}})(h_2 - h_1)
\]

From eqs 5.27-5.30 the air flow rate can be calculated if the mean air temperature of the cavity is known. This temperature has to be calculated with the heat flow equation of the cavity. So these equations
are coupled. A problem is the circulation of air in the cavity. Especially when the temperatures of the cavity surfaces are very different there can be much circulation and the presented temperature and ventilation will be but a rough estimate.

5.4.2 Heat flow in a cavity

Many constructions have ventilated cavities. The ventilation can be meant to get rid of moisture but also for heat (ventilated tropical roof). Moreover there are ventilated cavities in glazed constructions as solar air collector, air flow window, and double skin facade.

The most simple model of a cavity assumes 1D heat transfer in the cavity walls, perpendicular to the surface and constant surface coefficients. The model can be represented by a (linear) thermal network:

where: \( T_{s1}, T_{s2} \) = the surface temperatures 
\( T_{cav} \) = the mean air temperature in the cavity 
\( h_{cv1}, h_{cv2} \) = the surface coefficients for convective heat transfer 
\( h_r \) = surface coefficient for radiation (usually 5 W/m\(^2\)K but less with spectral selective coating e.g. 0.5 W/m\(^2\)K, see 6.5.1) 
\( R_1 \) and \( R_2 \) = the thermal resistances between the cavity surfaces and the exterior temperatures \( T_1 \) resp. \( T_2 \)
With the thermal resistance $R_V$ the heat flow added to the cavity air is calculated. This heat flow is according to the figure above:

$$\Phi = A \left( T_{\text{cav}} - T_{\text{in}} \right) / R_V$$

$A =$ cavity surface \hspace{1cm} (5.31)

This heat flow is also:

$$\Phi = \rho c_p V (T_{\text{out}} - T_{\text{in}})$$ \hspace{1cm} (5.32)

where: $T_{\text{out}} =$ the temperature of the air leaving the cavity
$T_{\text{in}} =$ the air temperature of the flow entering the cavity

So the thermal resistance $R_V$ is:

$$R_V = \frac{A}{\rho c_p V} \frac{T_{\text{cav}} - T_{\text{in}}}{T_{\text{out}} - T_{\text{in}}} = \frac{\gamma_T A}{\rho c_p V}$$ \hspace{1cm} (5.33)

If complete mixing is assumed then $\gamma_T = 1$. If there is no mixing (plug flow) and the velocity is very high the temperature gradient in the flow direction will be constant so $\gamma_T = 0.5$. These two situations are analogous to the situation in a room with completely mixed air (ventilation efficiency is 0.5) and with ideal displacement ventilation (efficiency = 1). The minimum efficiency $0$ i.e. the air that enters the room is leaving without taken up any heat ($\gamma_T = \infty$). So

$0.5 < \gamma_T < \infty$

If experimental data are lacking and the cavity is narrow mixing can be neglected and $\gamma_T$ can be assumed to be close to 0.5.

The air flow in ventilated cavities is a very complicated problem. To date not much progress is made yet with CFD. So the simple approach is still used. For understanding what happens as well as for design decisions it is often sufficient accurate. Experimental research however is indispensable on this topic.

- $h_v_1$ depends on velocity and temperature in a complicated way. By the ventilation $h_{v1}$ en $h_{v2}$ will be higher than without.
- In the cavity the air velocity will also depend on the coordinate normal to the cavity surface. The velocity at the cold side of the cavity will be lower than at the warm side and can have a reversed direction.
- The radiation exchange from a certain place to the opposite side of the cavity depends on the total temperature distribution and not just the spot temperature.
- There will be heat conduction also in the x-direction.
- In practice the flow is not uniform and not 2D. So what is the value for $\gamma_T$?
- Transient effects (e.g. by heat storage in walls) are always present.
The gradient factor $\gamma_T$ can be derived for a simple case:
- no temperature gradient perpendicular to the cavity surfaces
- the air velocity has everywhere in the cavity the same direction e.g. directed to the positive x-direction (see figure)
- the heat flow through the cavity surfaces is one-dimensional (perpendicular to the surface)
- the temperatures $T_1$ and $T_2$ and the heat sources $E_1$ and $E_2$ are independent of the x-coordinate.
- There are no transient effects
- The surface coefficients for radiative and convective heat transfer are constant.

The temperature of the incoming air at $x=0$ is $T_{in}$. The outgoing flow at $x=h$ has a temperature $T_{out}$. The mean cavity air temperature is $T_{cav}$.

If $b$ is the width and $d$ the depth and $v$ the (mean) air velocity then the heat flow density $q$ from the cavity surfaces to the cavity air follows from:

$$q_b \Delta x = \rho c_p b v d [T_{cav}(x + \Delta x) - T_{cav}(x)]$$

or

$$q = \rho c_p v d \frac{dT_{cav}}{dx}$$

If $A$ is the surface area of the cavity:

$$A = bh, \quad \bar{V} = vbd, \quad \rightarrow \quad vd = \frac{\bar{V}}{A}$$

The heat flow density can be calculated with the network (fig ). Solving is much easier if the delta network with the surface coefficients is transformed into a star network. If $h_1$ is the thermal conductance between surface 1 and the centre of the star, $h_2$ between surface 2 and the centre and $h_3$ between the cavity air temperature and the centre then:

$$h_1 = h_{cv1} + h_r(1 + \frac{h_{cv1}}{h_{cv2}}), \quad h_2 = h_{cv2} + h_r(1 + \frac{h_{cv2}}{h_{cv1}})$$

$$h_3 = h_{cv1} + h_{cv2} + \frac{h_{cv1} h_{cv2}}{h_r}$$

Let $U_1 = 1/(R_1 + 1/h_1), \quad U_2 = 1/(R_2 + 1/h_2)$

$$q = \frac{h_3 (U_1 + U_2)}{U_1 + U_2 + h_3} (T_\infty - T_{cav})$$

then

$$T_\infty = \frac{U_1 R_1 E_1 + U_2 R_2 E_2 + U_1 T_1 + U_2 T_2}{U_1 + U_2}$$

With this equation for $q$ the solution of the differential equation is:

$$T_{cav} = T_\infty - (T_\infty - T_{in}) \exp(-sx/h)$$

where:

$$s = \frac{A}{\rho c_p \bar{V} h_3 + U_1 + U_2}$$
With this solution the heat transfer by ventilation and through surface 1 and 2 can be calculated. The heat taken by the ventilation air in the cavity:

\[ \Phi = \rho c_p \dot{V} (T_{\text{out}} - T_{\text{in}}) = \rho c_p \dot{V} (T_{\infty} - T_{\text{in}}) (1 - \exp(-s)) \]

By the definition of \( R_V \) this heat flow is also

\[ \Phi = \frac{A}{R_V} (\bar{T}_{\text{cav}} - T_{\text{in}}) \]

\[ \bar{T}_{\text{cav}} = \frac{1}{h} \int_0^b T_{\text{cav}} \, dx = T_{\infty} + (T_{\text{in}} - T_{\infty}) \frac{(1 - \exp(-s))}{s} \]

With eq 5.33 the resistance \( R_V \) and \( \gamma_T \) can be calculated:

\[ R_V = \frac{1}{A} \left( \frac{1}{1 - \exp(-s)} - \frac{1}{s} \right) \frac{\dot{V}}{\rho c_p} = \frac{A}{\rho c_p \dot{V}} \gamma_T \]

### 5.4.3 Ventilated cavity wall

The traditional cavity wall was designed to keep moisture from the interior skin and moreover it was much more comfortable by the better U-value compared with the solid wall. Ventilation of the cavity should enhance the drying of the wall. It is still controversial if ventilation is necessary. With insulation of the wall ventilation can have a negative effect on the U-value.

In the figure below three cases are shown:

(i) Ventilation in a cavity at the exterior side of the insulation (correct way)

(ii) Ventilation at the interior side (wrong but might happen with bad workmanship)

(iii) Ventilation around the insulation and with outside (bad workmanship but not unrealistic)
Case i:

The network is shown in the figure. By the ventilation the exterior skin is more or less shortcircuited. This has not much influence on the total $U$ value as the insulation has the main contribution. As the temperature difference across the exterior blade is small also the ventilation will be small and the drying effect is expected to be small.

Case ii

Now the insulation is shortcircuited. This has a desastrous consequence on the $U$-value. As the cavity is at the warm side of the insulation there is a large temperature difference and much ventilation. This stresses the importance of not having a cavity at the warm side.

Case iii

The thermal network is shown in the figure. If there is no ventilation to the outside the air is circulating around the insulation material shortcircuiting the insulation. This is a very bad situation that is even worse with some supply of cold exterior air in the cavity. This bad workmanship happened often in the past.

5.4.4 An air collector

A simple air collector consists of a transparent cover plate a cavity and a black surface (absorber) with behind insulation material.

The amount of solar radiation absorbed by the absorber depends on the absorptivity $\alpha$, the transmittivity of the cover plate ($\tau$) and the reflectivity of the cover at the cavity surface ($\rho$) (see section 6.6):

$$E_2 = (\alpha \tau) E_{\text{sol}} = \frac{\alpha \tau}{1 - \rho (1 - \alpha)} \frac{E_{\text{sol}}}{E_{\text{sol}}}$$  \hspace{1cm} (5.34)

where $E_{\text{sol}}$ = the irradiance of the sun on the collector (W/m$^2$)

Let $T_1 = T_e$ and $E_1 = 0$, $U_2 = 0$ (ideal insulation). The heat transferred to the cavity air is:

$$\Phi = A \left( \frac{T_{\text{in}} - T_e}{R_v + 1/h_3 + 1/U_1} + \frac{(\alpha \tau) E_{\text{sol}}}{U_1} \right)$$  \hspace{1cm} (5.35)

where: $U = \text{loss coefficient}$

$F = \text{‘performance’ factor}$

The collector is delivering heat if the irradiance is above a certain threshold:

$$E_{\text{sol}} > \frac{U_1 (T_{\text{in}} - T_e)}{\alpha \tau}.$$  \hspace{1cm} (5.36)
So a low inlet temperature is important. Moreover the loss coefficient \( U_1 \) should be small: e.g. by a low emissivity coating on the absorber \( h_r \) will be small and the cover should have a high thermal resistance.

For a high performance factor \( F \) a high velocity is favorable, but the drawback is a low outlet temperature. Also a high \( h_{cv2} \) is good. This is ‘apparent’ higher when the absorber has a larger area (e.g. by ribbons). Usually the air doesn’t flow directly behind the cover plate but in air channels in the absorber to prevent contamination of the cover.

A special application of an air collector is the preheating of ventilation air in a balanced ventilation system. In that case \( T_{in} = T_c \) and the solar gain is:

\[
\Phi = AF (\alpha \tau) E_{sol} \tag{5.37}
\]

The total ventilation loss is:

\[
\Phi_n = \rho c_p \dot{V} (T_1 - T_c) - AF (\alpha \tau) E_{sol} \tag{5.38}
\]

The threshold disappeared. Yet a very large collector is useless as:

- By the lack of storage auxiliary heating is still necessary (e.g. at night). The air flow should not be larger than needed for the ventilation.

- The performance factor \( F \) decreases as \( \dot{V}/A \) decreases.

\[
\text{Cover plate of single glazing } R_2 = 1/25 \text{ W/ m}^2\text{K}, \quad \alpha \tau = 0.8
\]

\[
h_r = h_{cv1} = h_{cv2} = 5 \text{ W/ m}^2\text{K}, \quad \dot{V}/A = 50 \text{ m}^3/\text{m}^2\text{h}
\]

So \( h_1 = h_2 = h_3 = 15 \text{ W/m}^2\text{K}, \quad U_1 = 1/(1/25 + 1/15) = 75/8 \text{ W/ m}^2\text{K} \)

\[
s = \frac{A}{\rho c_p \dot{V}} \frac{h_1 U_1}{h_r + U_1} = \frac{3600 \cdot 15 \cdot 75/8}{1.2 \cdot 1000 \cdot 50 \cdot (15 + 75/8)} = 0.346
\]

\[
\gamma_T = 0.53
\]

\[
R_v = \frac{A\gamma_T}{\rho c_p \dot{V}} = \frac{3600 \cdot 0.53}{1.2 \cdot 1000 \cdot 50} = 0.032
\]

\[
F = \frac{1}{1 + (0.032 + 1/15) \times 75/8} = 0.52
\]

So \( \Phi = 0.42 \ A \ E_{sol} \)

If \( A = 4 \text{ m}^2 \) and \( E_{sol} = 500 \text{ W/m}^2 \) then \( \Phi = 840 \text{ W} \)
**5.4.5 Air flow window**

In an air flow window a window cavity is used as a ventilation exhaust duct or if the air is recirculated, the window is heated by the air to increase comfort.

Let $E_1 = E_2 = 0$, $T_1 = T_{in}$, $T_2 = T_e$ en $T_{in} = T_i$

Without solar radiation the heat flow density through the interior pane is:

$$q = \frac{U_1 U_2 (T_i - T_e)}{U_1 + U_2 + (1/(R_v + 1/h_3))} \quad (5.39)$$

Without flow the heat loss is

$$q \approx \frac{U_1 U_2 (T_i - T_e)}{U_1 + U_2}$$

So the heat loss is less as the ventilation air is cooled in the cavity and part of the heat diminishes the loss. The surface temperature increases so it is more comfortable.

The advantage of the airflow window is not the winter situation as insulation glazing nowadays makes this system superfluous. The main advantage is the possibility to put solar blinds in the cavity to prevent overheating, or to save cooling energy. As the blinds are protected it can be used for high-rise buildings. The ventilation air can come from interior or exterior and will leave to the outside.

The thermal network of this system is shown in the figure below.

![Diagram of airflow window](image)

It is a simplified model of a very complicated system. The real problem however is not the simplification but the uncertainty about input quantities as there are: the airflow rate, the surface coefficients at the cavity surfaces and at the solar protection, the optical properties of the different layers. Moreover there are many combinations possible with the HVAC installation.
Suppose the exterior glazing is double and interior pane is single.

\[
\frac{1}{R_1} = 7 \, \text{W/m}^2\text{K}, \quad \frac{1}{R_2} = 5 \, \text{W/m}^2\text{K}, \quad h_{cv1} = h_{cv2} = 4 \, \text{W/m}^2\text{K}, \\
h_1 = 5 \, \text{W/m}^2\text{K}, \quad \frac{\dot{V}}{A} = 25 \, \text{m}^3/\text{m}^2\text{h} \\
\text{So } h_1 = h_2 = 14 \, \text{W/m}^2\text{K}, \quad h_3 = 11.2 \, \text{W/m}^2\text{K} \\
U_1 = \frac{14}{3} \, \text{W/m}^2\text{K}, \quad U_2 = \frac{70}{19} \, \text{W/m}^2\text{K}, \\
\text{If } \gamma_T = 0.5 \text{ then } R_V = 3/(2 \times 25) = 0.06 \, \text{m}^2\text{K}/\text{W} \text{ and } \\
q = \frac{T_i - T_s}{3/14 + 19/70 + 3.19/(0.06+1/11.2)/14/70} = 1.14(T_i - T_c) \\
\text{Without the ventilation the heat loss is:} \\
q \approx 2.06(T_i - T_c) \\
\text{So there is less heat loss. Note that the position of the double glazing is not important. The inside is better to avoid condensation problems.}

5.4.6 A ventilated roof to avoid overheating

In order to avoid overheating by absorption of solar radiation by lightweight roof constructions a ventilated cavity in the roof can be used. The air can come from inside or from outside and the exhaust opening is at the highest place. This system is sometimes called a tropical roof and is used in warm climates.

Let \( T_{in} = T_e \), \( T_f = T_e + \alpha E_{sol}/h_e \)

\( T_{2f} = T_i \), \( E_2 = 0 \)

The mean heat flow density to the interior is:

\[
-q_2 = U_2 \frac{U_1(T_i + \alpha E_{sol}/h_e - T_i) + (T_e - T_i)/(R_V + 1/h_3)}{U_1 + U_2 + 1/(R_V + 1/h_3)}
\]

(5.40)

The heat flow density by solar radiation is:

\[
-q_2 = \frac{U_1U_2}{U_1 + U_2 + 1/(R_V + 1/h_3)}\alpha E_{sol}/h_e
\]

(5.41)

It is important to have a small \( R_V \). So this means that \( \dot{V}/A \) should be large. A deep cavity, a large vertical difference between the openings (large stack effect) the opening is such a position that the wind has a maximal effect.
5.5 Convective heat and vapour transfer through walls

A traditional method to prevent moisture problems in cavity structures is to ventilate the cavity with outdoor air (cavity wall, crawl space, warm roof). In the past with the non-insulated structures this was effective. Nowadays with the insulation material in the cavity the benefit is doubtful and ventilation might be the cause of many problems:

- Extra heat loss and interstitial condensation. Cold exterior air can flow through joints between insulation plates or e.g. through mineral wool plates to the warm side of the structure. For very well insulated structures this extra relative loss can be significant. More important however are the moisture problems (interstitial condensation).

- Surface condensation. By infiltration of the cold air the inside surface temperature can get below the dew point of the indoor air (the effect is comparable with the thermal bridge effect)

- Corrosion of metal in the cavity. By the insulation the cavity is much colder and the relative humidity is higher. This might cause problems. The reason not much damage has been observed yet might be that corrosion is a very slow process.

If the indoor pressure is higher than the outdoor one, humid indoor air can enter the structure (exfiltration) causing interstitial condensation. This can also happen in non-ventilated structures but ventilation aggravates it. So the supposed ‘solution’ can be the cause. Also openings in the cladding that decrease the wind load (pressure equalisation) can be the cause of more exfiltration if the interior layer is not airtight (e.g. in timber frame houses).

In practice interstitial condensation caused by exfiltration is the main moisture problem as indoor air is more humid than outdoor air. Structures that are safe according to Glaser appear to have problems due to airflows. The performance requirements put on the structure regarding air tightness are not always realistic. It is also very difficult to predict pressure differences across the structure or the path of the airflow. Additionally problems always appear at complicated 3-dimensional details (joints, thermal bridges, etc).

By temperature differences in an air permeable insulation material the air inside the material can move and the insulation value can be much less than expected.

5.5.1 Convective heat transfer through materials

We shall restrict ourselves to the simple steady state 1D-situation where air flows through a porous structure. The equation for the pressure is similar to the steady state heat conduction equation and was treated already in section 1.3.4. Equation 1.26 for one-dimensional steady state air transport is:
The air permeability $k_a$ is a parameter of the pore system. At higher moisture content the permeability will be smaller. In materials with only closed pores the permeability is zero. Most building materials have a low permeability. An important exception is mineral wool, which has a rather high permeability.

If the permeability $k_a$ is constant the pressure is a linear function of place. The density of the mass flow rate $g_a = -k_a dp/dx$ through a layer with thickness $d$ and surface area $A$ is:

$$
g_a = \rho_a \frac{\dot{V}}{A} = k_a \frac{\Delta p}{d} \tag{5.42}
$$

where: $\dot{V}$ = the airflow rate [m$^3$/s]  
$\Delta p = p_1 - p_2$ [Pa]

The quantity $k_a/d$ is sometimes called the air permeance.

Mineral wool has a permeability $k_a = 29 \cdot 10^{-6}$ (s), $\lambda = 0.04$ W/mK, $d = 10$ cm. Suppose the pressure difference is $\Delta p = 2$ Pa

$$
\frac{\dot{V}}{A} = k_a \frac{\Delta p}{\rho_a d} \rightarrow \dot{V} = 29 \cdot 10^{-6} \cdot \frac{2}{1.2} = 48 \cdot 10^{-5} \text{ m}^3/\text{m}^2\text{s}
$$

If air flows through a permeable material (e.g. mineral wool, material with open cells) and one assumes that this air has the same temperature as the surrounding material, the density of the heat flow rate in the direction of the flow is according to eq.1.27:

$$
q = \rho_a c_p \frac{\dot{V}}{A} \frac{T - \Delta T}{\lambda} = \rho_a c_p \frac{\dot{V}}{A} R \tag{5.43}
$$

For a steady state flow this yields:

$$
\frac{dq}{dx} = 0 \text{ or } q = \text{constant}
$$

With $x = 0$: $T = T_1$ and $x = d$: $T = T_2$ the solution is:

$$
T = T_1 - \frac{\exp{(sx/d)} - 1}{\exp(s) - 1} (T_1 - T_2) \quad \text{with:} \quad s = \frac{c_p k_a}{\lambda} = \rho_a c_p \frac{\dot{V}}{A} R \tag{5.44}
$$

where: $R = d/\lambda$, the thermal resistance of the layer.  
$s = \text{the Péclet number; the ratio of convective and conductive heat transfer.}$

If there are several layers $R$ is just the total resistance without convection. The heat flow rate through the construction is:
If the airflow were parallel to the layer instead of uniformly through the layer the resistance $R_{cv}$ would be equal to $R$. So if the flow path is not known the value is somewhere between $R$ and $R_{cv}$.

The Peclet number of the mineral wool example above is $s = 2.5 \times 48.10^{-5} \times 1200 = 1.44$. The resistance $R_{cv}$ is:

$$R_{cv} = A \frac{\exp(s)-1}{\rho \varepsilon c_p \dot{V}} R > R$$

(5.45)

The same resistance is found with the same air flow through a gap in 22.5cm of mineral wool instead of 10cm. The insulation performs like a heat exchanger.

Suppose it is cold outside and the ventilation flows through the insulation to the outside (exfiltration). The hot humid air will cool down in the insulation recovering part of the heat but also increasing the risk of interstitial condensation in the insulation material.

If the air flows to the interior (infiltration) then with the same external climate the heat flow is:

$$\Phi = \rho c_p \dot{V} T_e + A \frac{T_e - T_i}{R_{cv}} = \left( \rho c_p \dot{V} + \frac{A}{R_{cv}} \right) (T_e - T_i) + \rho c_p \dot{V} T_i$$

So if this air flow is needed for ventilation the preheating by the insulation material has a positive effect. It is sometimes called active insulation. If this air is not needed, one can see from the second part of the equation that the effect is negative.

$$\left( \frac{\rho c_p \dot{V}}{A} + \frac{1}{R_{cv}} \right) = 1.2 \cdot 0.48 + 1/5.8 = 0.75 = \frac{1}{1.34} \text{W/m}^2\text{K}$$

The effective resistance corresponds to 5 cm thermal insulation. The interior surface temperature is lower by the cold infiltration so that transmission loss but also risk of surface condensation increases.
5.5.2 Convective vapour transfer through materials

The equation for the vapour flow density is analogous to the one for heat:

\[
G = 0.62 \cdot 10^{-5} \rho_a \dot{V}_p - A \frac{\delta_a}{\mu} \frac{dp}{dx} \tag{5.46}
\]

So the solution is similar:

\[
p = p_1 - \frac{\exp(s_x/d) - 1}{\exp(s_y) - 1} (p_1 - p_2) \tag{5.47}
\]

\[
s_v = \frac{0.62 \cdot 10^{-5} \Delta p \kappa \mu}{\delta_a} = \frac{0.62 \cdot 10^{-5} \rho_s \dot{V} \mu d}{A \delta_a} = \frac{0.62 \cdot 10^{-5} \rho_s \dot{V}}{A} Z
\]

\[
G = 0.62 \cdot 10^{-5} \rho_s \dot{V} p_1 + \frac{(p_1 - p_2)}{Z^{cv}}
\]

The apparent vapour resistance is:

\[
Z^{cv} = A \frac{\exp(s_y) - 1}{0.62 \cdot 10^{-5} \rho_s \dot{V}} \frac{\exp(s_y) - 1}{s_v} Z \tag{5.48}
\]

Note that the vapour pressure is a non-linear function of \(x\).

The correction for vapour is much larger than for heat as:

\[
\frac{s_v}{s} = \frac{0.62 \cdot 10^{-5} \mu \lambda}{c_p \delta_a} = 34 \mu \lambda >> 1 \tag{5.49}
\]

The smaller the porosity of the material is (large \(\lambda\) and \(\mu\)) the larger is this fraction. In other words when the effect of airflow on the heat flow can be neglected (small \(s\)) it doesn’t mean that this is true for the vapour flow. By the fact that \(sv > s\) it is possible that interstitial condensation occurs within a material layer. If air flows from the warm side to the cold side the temperature in the material decreases slightly different from linear with depth. The vapour pressure however will decrease less due to the higher \(sv\) and can rise above saturation pressure. See figure.

The temperature at a certain place in the construction can be calculated with the continuity of heat flow density. Suppose the heat resistance between an interface and exterior is \(R_{se}\) and between this interface and interior \(R_{si}\). The temperature of the interface is:

\[
\rho c_p \dot{V} T_i + \frac{A}{R_{si}} (T_i - T_s) = \rho c_p \dot{V} T_s + \frac{A}{R_{se}} (T_s - T_e)
\]

\[
T_s = T_i + \frac{1/R_{se}}{1/R_{si} + \rho c_p \dot{V}/A + 1/R_{se}} (T_e - T_i) \tag{5.50}
\]
Air flow has a very large effect on interstitial condensation. For the calculation of interstitial condensation (see section 4.5.4) the temperature at the critical layer must be known and the thermal and vapour resistances at both sides.

Suppose that the thermal resistance without airflow from the critical interface to outdoors is $R_{ce}$ and to indoors $R_{ci}$. The critical temperature with flow follows from eq. 5.50:

$$T_c = \frac{(1+s_e)/R_{cv}^{ce}}{(1+s_e)/R_{cv}^{ci} + 1/R_{cv}^{ce}} T_i + \frac{1/R_{cv}^{ce}}{(1+s_e)/R_{cv}^{ci} + 1/R_{cv}^{ce}} T_{cv}$$

The corrected thermal resistances are:

$$R_{cv}^{ce} = R_{ce} \exp(s_e) - \frac{1}{s_e}$$

$$R_{cv}^{ci} = R_{ci} \exp(s_i) - \frac{1}{s_i}$$

(5.51)

with $s_e = R_{ce} \rho_a c_p V/A$ and $s_i = R_{ci} \rho_a c_p V/A$

In de same way the vapour pressure must be calculated at the critical layer. If it turns out that this pressure is higher than the saturation pressure at the temperature of the critical layer condensation occurs. The moisture resistances $Z_{ce}$ and $Z_{tot}$ must be corrected in the same way as the thermal resistances:

$$Z_{cv}^{ce} = Z_{ce} \exp(s_{ve}) - \frac{1}{s_{ve}}$$

$$Z_{cv}^{ci} = Z_{ci} \exp(s_{vi}) - \frac{1}{s_{vi}}$$

(5.52)

with $s_{ve} = 0.62 \cdot 10^{-5} Z_{ce} \rho_a V/A$ and $s_{vi} = 0.62 \cdot 10^{-5} Z_{ci} \rho_a V/A$

The amount can be calculated with eq.4.33. With air flow this is:

$$g = \frac{(1+s_{vi})}{Z_{cv}^{ci}} (p_i - p_{sat}(T_i)) - \frac{(1/Z_{cv}^{ce}) (p_{sat}(T_c) - p_e)}$$

(5.53)

A cavity wall is insulated with 8 cm mineral wool. The cavity surfaces are not plastered. Condensation might occur at the inside of the exterior skin. As a matter of fact this is not a real problem as the exterior skin can stand that and in summer it will dry easily.

**Example**

<table>
<thead>
<tr>
<th>$21^\circ$C; $\Theta_e = 2.5^\circ$</th>
<th>The thermal resistances at either side of the critical layer are: $R_{ce} = 0.14$ m$^2$K/W; $R_{ci} = 2.4$ m$^2$K/W; $\mu d_e = 1.5$ m; $\mu d_i = 1.6$ m; $V/A = 5 \cdot 10^{-5}$ m$^3$/m$^2$s; $\Theta_i = C$; $p_i = 936$ Pa; $p_e = 600$Pa;</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_i = 0.144$, $s_e = 0.0084$, $s_{vi} = 3.264$, $s_{ve} = 3.06$</td>
<td>$R_{cv}^{ce} = 1.004 - 0.14$ m$^2$K/W; $R_{cv}^{ci} = 1.076 - 2.4$ m$^2$K/W</td>
</tr>
<tr>
<td>$Z_{cv}^{ce} = 6.63 \cdot 1.5/\delta_a$</td>
<td>$Z_{cv}^{ci} = 7.07 \cdot 1.6/\delta_a$</td>
</tr>
</tbody>
</table>

The corrections for the moisture resistances are very large.
\[ T_c = 3.59 \, ^\circ C, \quad p_{sat}(T_c) = 791 \, Pa, \quad p_c = 921 \, Pa > p_{sat}(T_c) \]

There is interstitial condensation. \( g = 30.7 \delta_a \, \text{kg/m}^2\text{s} \)

Without air flow: \( T_c = 3.52 \, ^\circ C \) and \( p_{sat} = 786 \, Pa \) \( p_c = 763 \, Pa < p_{sat} \)

So no condensation.

The next question is: is there condensation when the same amount of air crosses the construction but in a different way. One can distinguish 4 cases of air flow:

i) through-through: Uniform flow through interior and exterior skin (this is done in the example above)

ii) through-pass: Uniform through interior skin and through vent holes in the exterior skin.

iii) pass-through: through cracks of the interior skin and uniform through the exterior skin (not likely)

iv) pass-pass: through cracks of the interior skin and through vent holes in the exterior skin.

With this example the effect of different exfiltration paths is shown. In all cases it is assumed that at the critical layer the temperature of the flowing air is equal to the temperature of the layer.

The results are summarized in the table below.

<table>
<thead>
<tr>
<th>Exfiltration paths</th>
<th>Rce</th>
<th>Zce</th>
<th>Through/Pass</th>
<th>Tc</th>
<th>psat(Tc)</th>
<th>g</th>
<th>pc</th>
</tr>
</thead>
<tbody>
<tr>
<td>Through</td>
<td>0.141</td>
<td>10.28</td>
<td>Through</td>
<td>2.581</td>
<td>12.75</td>
<td>3.59</td>
<td>791</td>
</tr>
<tr>
<td>Pass</td>
<td>0.14</td>
<td>1.5</td>
<td>Pass</td>
<td>2.4</td>
<td>1.6</td>
<td>3.59</td>
<td>791</td>
</tr>
</tbody>
</table>

- The air flow is the cause of condensation.
- Most condensation occurs if the airflow reaches the critical layer by a crack or vent hole. In practice this will be worse as the condensation surface is close to the crack (so local much condensation).

It is recommended to base the calculation on the last assumption and to neglect the influence of the air flow on the temperature (worst case). Especially for vapour retarders at the room side of the wall the air flow plays an important role. An estimate has to be made of the unavoidable small holes and overlaps of the used foil.
Derive the equations for the force on a bend.

solution: The mass flow is: \( G = A_1 \rho_1 v_{in} = A_2 \rho_2 v_{out} \)

So the force on a bend in a tube follows from:

\[
G(-v_{in}) = A_1(p_1 - p_\infty) + F_x
\]

\[
G(v_{out}) = -A_2(p_2 - p_\infty) + F_y
\]

In a room there is a constant vapour production of 1.2 kg/h. The ventilation is constant and equal to 180 m³/h.

a. How much is the increase of the humidity ratio of the indoor air?
(answer: 1200/1.2/180=5.55 gr/kg)

b. For comfort reasons it is decided to ventilate 90% of the time only 100 m³/h and 10% of the time a higher ventilation in such a way that the mean ventilation is again 180 m³/h. What is in this case the mean humidity ratio of the room?
(answer: 0.9x100+0.1xV =180 \( \Rightarrow \) V =900 m³/h (!). The mean increase of the humidity ratio is 0.9x1200/1.2/100+0.1x1200/1.2/900 = 9.11 gr/kg.)

c. What can be concluded from a and b?
( answer: the very high ventilation of 900 m³/h has no effect. Even with a factor 10 higher the effect is very small. When the vapour production is constant, a constant ventilation is the most effective and energy efficient. In a bathroom after a shower the vapour production goes on by the wet surfaces. Also the ventilation should go on until it is completely dry!)
6 Radiation

6.1 Introduction

Radiation is the propagation of electromagnetic waves. All forms of matter emit and absorb radiation dependent on the temperature. This radiation that is related to the thermal movement of molecules is called thermal radiation. By emitting radiation a body looses energy (heat). If this emitted radiation hits another body a fraction can be absorbed and this part will be transformed into heat. This body will also emit radiation back to the first one, so heat is exchanged between the two bodies. We can feel that if we stand for a cold window or stand for a hot stove. The heat transfer by thermal radiation requires no matter and in building physics it is primarily between surfaces. Radiation is a surface phenomenon.

Waves have a wavelength (\(\lambda\)), a frequency (f) and a propagation velocity (v). The relation between these properties is:

\[\lambda = \frac{v}{f}\]

Electromagnetic waves propagate along straight lines. In vacuum the velocity is \(3 \times 10^8\) m/s, the light velocity. In air the velocity can be considered equal to the light velocity but in fluids and solids the velocity is less.

If the radiation is characterized with a wavelength instead of a frequency it is usually the wavelength in vacuum. The unit can be nm \((10^9\) m), \(\mu\)m \((10^6\) m) or Å \((10^{10}\) m). For building physics three intervals are important: ultraviolet radiation \((4\) nm < \(\lambda\) < 380 nm), visible radiation (light) \((380\) nm < \(\lambda\) < 780 nm) and infrared radiation \((780\) nm < \(\lambda\) < 100,000 nm). Except these properties polarization of the waves is important but this will not be treated here.

<table>
<thead>
<tr>
<th>radiant flux</th>
<th>0.02</th>
<th>0.38</th>
<th>0.782</th>
<th>100(\mu)m</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Phi)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ultraviolet</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>visible</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>infrared</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>thermal radiation</td>
<td>0.1</td>
<td></td>
<td></td>
<td>100(\mu)m</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The radiant flux \(\Phi\) (W) is the power emitted, transferred or received as radiation. The radiant flux incident on a surface per m\(^2\) of the surface area is called the irradiance \(E\) (W/m\(^2\)) of that surface (for light this is the illuminance).
The radiant flux emitted by a surface element divided by the area of that element (the surface is the radiation source) is called the (self) radiant exitance (or radiant emissive power) $M$ (W/m$^2$).

Radiation can be a parallel beam but more often it is coming from different directions distributed in space. The radiant flux leaving a surface per solid angle and per m$^2$ perpendicular to the considered direction is called the radiance $L$ (W/m$^2$sr) (the corresponding property for light is luminance).

In a spherical coordinate system the directional dependence can be given by the angles $(\Theta, \phi)$ (the dependency on the place in a global coordinate system is skipped for convenience):

$$L = L(\Theta, \phi)$$

So the irradiance on a point on a surface by radiation with an angular distribution $L$ directed to that point is:

$$E = \frac{2}{\pi} \frac{\cos \Theta}{\Omega} \left[ \int_0^{\pi/2} \int_0^{2\pi} L \cos \Theta \sin \Theta \, d\Theta \, d\phi \right]$$  \hspace{1cm} (6.1)

Where: $\Theta = \text{angle between the de normal on the surface and the considered angle}$

$\Omega = \text{solid angle} \left( d\Omega = \sin \Theta d\Theta d\phi, \text{hemisphere is } 2 \pi \text{ sr} \right)$

So if $L$ is independent of $\phi$ and $\Theta$: $E = \pi L$

Note: If the radiation with radiance $L$ is leaving the surface, $E$ must be replaced by $M$ (the exitance)

If radiation has but one wavelength it is called monochromatic. Usually it is composed of radiation with different wavelengths; it has a spectrum. If the radiance with wavelengths between $\lambda$ and $\lambda + \Delta \lambda$ equals $\Delta L$ then the spectral radiance is defined as:

$$L_\lambda = \lim_{\Delta \lambda \to 0} \frac{\Delta L}{\Delta \lambda}$$

and

$$L = \int_{\lambda_1}^{\lambda_2} L_\lambda \, d\lambda \left( \text{W/m}^2\text{sr} \right)$$

If $L_\lambda$ is a 'smooth' function of $\lambda$, the spectrum is called continuous. The function can also have 'gaps' ($L_\lambda = 0$). In that case it is a band spectrum or if the bands are very narrow it is a line spectrum.

In the same way the spectral exitance and spectral irradiance can be defined.
Suppose the sky is uniform, (isotropic diffuse solar radiation). How much is the irradiance on an inclined surface, slope \( \beta \), as a fraction of the irradiance on a horizontal surface?

The irradiance on a horizontal surface is:

\[
E_h = \frac{L \cos \Theta \sin \Theta}{\pi} \int \cos \sin d\Theta = \frac{L}{\pi}
\]

Of the sloped surface:

\[
E_\beta = L \int \phi f(\phi) \cos \Theta \sin \Theta d\Theta + L \int \phi \cos \Theta \sin \Theta d\Theta
\]

\[
E_\beta = 0.5L \int_0^{\pi/2} \phi (\sin(f(\phi)))^2 + 0.5\pi L
\]

\[
\tan(\beta) = \frac{Z}{x} = \frac{\cos \Theta}{\sin \Theta \cos \varphi} \Rightarrow f(\phi) = \cot(\cos \varphi \tan \beta)
\]

So

\[
E_\beta = 0.5L \int_0^{\pi/2} \frac{d\phi}{1 + (\tan \beta \cos \varphi)^2} + 0.5\pi L
\]

The result is:

\[
E_\beta = 0.5(1 + \cos \beta)E_h
\]
The radiant flux $\Phi$ leaving a surface element with area $dA_1$ and received by a face with area $dA_2$ is:

$$\Phi = L dA_1 \cos \Theta_1 d\Omega_1$$

where $\Theta_1$ = polar angle of local coordinate system on the surface 1  
$L$ = radiance of radiation leaving surface 1  
$d\Omega_1$ = solid angle containing surface $dA_2$

d$\Omega_1$ equals:

$$d\Omega_1 = \frac{dA_2 \cos \Theta_2}{r_{12}^2}$$

where $r_{12}$ = the distance between the surfaces.  
$\Theta_2$ = polar angle of local coordinate system on the surface 2

So:

$$\Phi = L \frac{dA_1 \cos \Theta_1 dA_2 \cos \Theta_2}{r_{12}^2}$$

### 6.2 Surface emission

#### 6.2.1 Blackbody emitter

A body that absorbs all radiation is called a blackbody. In nature ideal blackbodies don’t exist, but for a limited range of wavelengths it can be close. An example is an enclosure with a very small hole. The radiation incident on that hole from the outside of the enclosure is completely trapped by the enclosure and only a very small part of this incident radiation will leave the enclosure by the hole i.e. the hole is close to a blackbody emitter.

The emitted radiation of a blackbody is the maximum that can be emitted at the temperature of the body. The equation of the emitted spectral radiance of a blackbody is known as Planck’s distribution:

$$L_{bl}(\lambda, T) = \frac{C_1}{\pi \lambda^5 \exp (C_2 / \lambda T) - 1}$$  \hspace{1cm} (6.2)

with:  
$C_1$ = first radiation constant: $3.742 \cdot 10^{-16}$ W/m$^2$  
$C_2$ = second radiation constant: $1.439 \cdot 10^{-2}$ mK  
$T$ = absolute temperature (K) of the body

A blackbody is also called a Planckian body.
The spectral radiance has a maximum. The wavelength at which the maximum occurs is the solution of the equation \( \frac{dL_\lambda}{d\lambda} = 0 \). The result is:

\[
\lambda = \lambda_{\text{max}} = 2.8976 \cdot 10^{-3} / T
\]

So the wavelength at which the spectral emissive power is maximum will shift to shorter values at higher absolute temperatures. This is called Wien’s displacement law.

\[\lambda_{\text{max}} \approx 10^{-5} \text{m} = 10,000 \text{ nm}, \text{ so in the infrared part of the spectrum.}\]

E.g. at 290K \( \lambda_{\text{max}} \approx 10^{-5} \text{m} = 10,000 \text{ nm}, \) so in the infrared part of the spectrum. The radiation emitted by surfaces with temperatures between ca. 260 and 350 K is called longwave radiation (\( \lambda > 3000 \text{ nm} \)). For visible radiation a much higher temperature is needed e.g. the blackbody temperature of solar radiation is \( T \approx 6,000 \text{ K} \). Solar and visible radiation (light: \( 3000 \text{ nm} > \lambda > 3 \text{ nm} \)) are shortwave radiation.

The emitted radiation of a blackbody is independent of the direction. A surface for which the emitted radiance is independent of the direction is named a Lambertian surface or diffuse emitter. The emitted radiant flux is distributed angularly according to \( L_\Omega \cos \Theta \). This is called Lambert’s cosine law.

The spectral exitance can easily be derived:

\[
M_{b\lambda} = \int L_{b\lambda} \cos \Theta d\Omega = \pi L_{b\lambda}
\]

The exitance of a Planckian emitter is:
\[ M_b = \int_0^\infty N_{b\lambda} \, d\lambda = \sigma T^4 \]  
(6.4)

where: \( \sigma = 5.67 \cdot 10^{-8} \text{W/m}^2\text{K}^4 \): Stefan-Boltzmann constant.

This simple formula is named the Stefan-Boltzmann law.

The sun can be considered as a Planckian emitter with a temperature of approximate 5800K.

So maximum spectral exitance is at \( \lambda_{\text{max}} \approx 2.8976 \cdot 10^{-3}/5800 = 500 \) nm, close to the maximum of the photopic spectral luminous efficiency of the eye. That is not a coincidence.

The radius of the sun is about \( R_{\text{sun}} = 0.7 \cdot 10^6 \text{ km} \) and the distance to the earth is \( R_{\text{sun-earth}} = 150 \cdot 10^6 \text{ km} \). So the solar rays that hit a place on the earth are within a solid angle of:

\[ d\Omega = \pi \left( \frac{R_{\text{sun}}}{R_{\text{sun-earth}}} \right)^2 = 7 \cdot 10^{-5} \text{ radians}. \]

So this is a very very small angle. For this reason the solar rays received by the earth can be considered as parallel.

The irradiance on a surface perpendicular to the beam just at the outside of the atmosphere (extraterrestrial) can also be calculated:

\[ E_o = \sigma T^4 \left( \frac{R_{\text{sun}}}{R_{\text{sun-earth}}} \right)^2 = 1390 \text{ W/m}^2 \]

The value of 1390 W/m² is not very exact, more exact is 1364 W/m² (as the distance between the sun and the earth has a cyclic annual variation it is also not a constant). It is named however the solar constant.
6.2.2 Diffuse grey emitters

The emitted radiation of a surface element can be written as:

\[ L_s = \varepsilon(\Theta, \varphi, \lambda, T_s) L_{\text{bs}}(\lambda, T_s) \]

where \( \varepsilon = \) the emittance (emissivity).

In building physics \( \varepsilon \) (for longwave radiation) is usually approximated by a constant. If the emittance is a constant < 1 the emitter is named diffuse grey.

Grey means that the emittance is supposed to be independent of the wavelength in the wavelength range of interest (here longwave irradiation).

So: \( \varepsilon = \varepsilon(\Theta, \varphi, T_s) \). Most surfaces can be considered as isotropic (no azimuthally dependence), so: \( \varepsilon = \varepsilon(\Theta, T_s) \) and \( L = \varepsilon(\Theta, T_s) \sigma T_s^4 / \pi \). For the exitance a hemispherical emittance can be introduced:

\[
M = \varepsilon_h(T_s) \sigma T_s^4, \quad \varepsilon_h(T_s) = 2 \int_0^\pi \varepsilon(\Theta, T_s) \cos \Theta \sin \Theta d\Theta
\]

Diffuse means that the emittance is assumed to be independent of the direction of the emitted radiation (Lambertian emitter). \( \varepsilon = \varepsilon(T_s) \) and \( M = \varepsilon(T_s) \sigma T_s^4 \)

For diffuse emitters \( \varepsilon_h = \varepsilon \). The deviation from 1 of the ratio \( \varepsilon(0,T_s)/\varepsilon_h \) shows the validity of the diffuse assumption. For electrical non-conducting smooth surfaces this ratio is about 0.95. If this surface is rough (most of them are in practice) the value is higher: 0.98. For polished metal surfaces the ratio is about 1.2.

The picture at the left shows \( \varepsilon(\Theta,T_s) \): dotted line lambertian, right stony material, left polished metal

Moreover in practice also the temperature dependence of the emittance is not taken into account. For heat exchange in building physics the temperatures are within the range of -20°C and 80°C. This is a small range in terms of absolute temperature.

The assumption of a constant emittance for longwave radiation is enough accurate for practice if the emittance is \( \varepsilon > 0.85 \). This is true for most building materials. Materials that conduct electricity very well e.g. metals, have often a low emittance and are if polished surely no grey emitters. Also low emittance coatings on glazing are not grey.

A problem for low emittance surfaces is the reliability of the value. This value is measured with clean surface and often at a lower surface relative humidity (high temperature) than relevant for the application. Absorbed water molecules can raise the value at a higher relative
humidity and contamination (pollution) of the surface after years might raise the emittance even more.

One should realise that the grey body assumption and the lack of detailed accurate information limits the accuracy of the complicated radiation exchange calculations in building simulation models.

<table>
<thead>
<tr>
<th>surface</th>
<th>ε</th>
<th>surface</th>
<th>ε</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polished gold/silver/copper</td>
<td>0.02</td>
<td>Glass/porcelain</td>
<td>0.92</td>
</tr>
<tr>
<td>Oxidised copper</td>
<td>0.78</td>
<td>Brick, plaster</td>
<td>0.93</td>
</tr>
<tr>
<td>Aluminium, polished</td>
<td>0.05</td>
<td>Wood</td>
<td>0.90</td>
</tr>
<tr>
<td>Aluminium, oxidised</td>
<td>0.30</td>
<td>Roofing felt</td>
<td>0.92</td>
</tr>
<tr>
<td>Steel, hot rolled</td>
<td>0.77</td>
<td>Paint, black lack</td>
<td>0.97</td>
</tr>
<tr>
<td>Steel, red rusted</td>
<td>0.61</td>
<td>Paint, oil based</td>
<td>0.94</td>
</tr>
<tr>
<td>Steel, zinc layer</td>
<td>0.26</td>
<td>Paint, white</td>
<td>0.85</td>
</tr>
<tr>
<td>Steel, polished</td>
<td>0.27</td>
<td>Marble, polished</td>
<td>0.55</td>
</tr>
<tr>
<td>Lead, oxidised</td>
<td>0.28</td>
<td>Paper</td>
<td>0.93</td>
</tr>
<tr>
<td>water</td>
<td>0.95</td>
<td>Ice °C</td>
<td>0.97</td>
</tr>
</tbody>
</table>

Some values of the emittance

6.2.3 Absorption

Surfaces also absorb radiation. The ratio of the spectral radiance absorbed to that of the incident one is called the absorptance $\alpha$. The absorptance is dependant on the material properties of the surface, its temperature and the direction and wavelength of the incident radiance.

For any surface the absorptance is exactly the same function of wavelength, direction and temperature as the emittance. This is called Kirchhoff’s law. As a consequence of this law the absorptance of a blackbody (emittance is 1) equals 1 i.e. all radiation is absorbed. That’s why a Planckian emitter is called a blackbody emitter.

The emittance for longwave radiation is usually assumed a constant, so then also the absorptance is a constant

$$\alpha = \varepsilon. \quad (6.6)$$

In building physics the emitted radiation is longwave but the absorbed radiation can be longwave and shortwave as well. Only for longwave Kirchhoff’s law is relevant (shortwave exitance is at very high temperatures!):

$$\alpha \text{ (longwave)} = \varepsilon \text{ (longwave)}$$

$$\alpha \text{ (shortwave)} \neq \varepsilon \text{ (longwave)}$$

E.g. white paint can have an emittance $\varepsilon = 0.9$ and an absorptance for solar radiation $\alpha = 0.25$. The longwave radiative properties of a surface are not visible (visible means wavelengths of light, so shortwave).
Some values for shortwave absorptance

<table>
<thead>
<tr>
<th>material</th>
<th>$\alpha$</th>
<th>material</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>snow</td>
<td>0.15</td>
<td>Concrete, faded plants light flooring</td>
<td>0.6-0.7</td>
</tr>
<tr>
<td>White paint</td>
<td>0.25</td>
<td>Leaves and grass</td>
<td>0.75</td>
</tr>
<tr>
<td>Whitewashed surface</td>
<td>0.30</td>
<td>Dark flooring and carpets</td>
<td>0.8-0.9</td>
</tr>
<tr>
<td>Light colours</td>
<td>0.3-0.5</td>
<td>Humid soil</td>
<td>0.9</td>
</tr>
<tr>
<td>polished aluminium</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Brick, yellow</td>
<td>0.55</td>
<td>Slate, dark grey</td>
<td>0.9</td>
</tr>
<tr>
<td>Brick, red</td>
<td>0.75</td>
<td>bitumen</td>
<td>0.93</td>
</tr>
</tbody>
</table>

For any surface the absorptance equals the emittance. This is called Kirchhoff’s law:

$$\alpha(\Theta, \phi, \lambda, T) = \varepsilon(\Theta, \phi, \lambda, T)$$

so the hemispherical $\alpha$ is:

$$\alpha_h = \int \int L_{\lambda, \omega} \alpha(\Theta, \phi, \lambda, T) \cos \Theta \ d\Omega \ d\lambda / E_i$$

Kirchhoff’s law is a consequence of the second law of thermodynamics:
the heat flow between two surfaces with the same temperature must be zero.

Proof: Suppose there are two small surface elements $dA_1$ and $dA_2$ of different material but with the same temperature. The faces exchange heat by radiation. According to the second law of thermodynamics there is no net heat flow from surface 1 to 2., or the heat flow from surface 1 to 2 must be equal to the flow from 2 to 1, so:

$$dA_1 \cos \Theta_1 L_{\lambda_1} d\Omega_1 \alpha_2 = dA_2 \cos \Theta_2 L_{\lambda_2} d\Omega_2 \alpha_1$$

The solid angles $d\Omega_1$ and $d\Omega_2$ are:

$$d\Omega_1 = \frac{dA_2 \cos \Theta_2}{r_{12}^2}, \quad d\Omega_2 = \frac{dA_1 \cos \Theta_1}{r_{12}^2}$$

where $r_{12} = $ the distance between the surfaces.

So:

$$L_{\lambda_1} \alpha_2 = L_{\lambda_2} \alpha_1 \rightarrow \frac{L_{\lambda_1}}{\alpha_1} = \frac{L_{\lambda_2}}{\alpha_2} = \frac{L_{\lambda_2}}{\alpha_1}$$

So $L/\alpha$ is independent of the material; i.e. is a universal function. If $\alpha=1$ it is a blackbody so the emitted radiation $L_\lambda = L_{\lambda b}$ or for any other material $L/\alpha = L_{\lambda b}$. This means $\varepsilon = \alpha$.

As the grey emitter assumption is not exact the hemispherical $\alpha$ is not exactly equal to the hemispherical $\varepsilon$. If the emitter is assumed diffuse grey one has to use $\varepsilon = \alpha$ in order not to violate the second law of thermodynamics.
6.3 Reflection and transmission

When radiation hits a surface a fraction will be reflected. The reflected radiant flux is of course less than the incident one but the radiation is also scattered; the angular distribution is different. Similarly, the radiation that is transmitted through a transparent or translucent material can be partially absorbed and scattered. Especially for shortwave radiation the wavelength dependency (colour) of absorption and scattering is important.

The objects in our environment emit hardly any shortwave radiation. That we can see them is because they reflect the shortwave radiation from the sun and the artificial lighting.

The fraction of the incident irradiance $E$ that is reflected by a surface is the reflectance $\rho$ and the fraction that is transmitted is the transmittance $\tau$.

So:

$$\alpha + \rho + \tau = 1$$  \hspace{1cm} (6.7)

6.3.1 Longwave radiation

Most solids and fluids are opaque for longwave radiation ($\tau = 0$). E.g. glass is opaque for longwave and transparent for shortwave radiation. Exceptions are some plastics (polyethylene) en quarts. For longwave radiation almost always the relation below yields:

$$\varepsilon = \rho - 1 = \alpha$$  \hspace{1cm} (6.8)

So there is no need to use the symbols $\alpha$ en $\rho$ for longwave, $\varepsilon$ is enough. Unless stated otherwise $\alpha$ en $\rho$ will only be used for shortwave radiation.

The sum of the (direct) emitted radiation and the reflected radiation is called the (radiant) exitance. The old name is radiosity.

With thermographic cameras the exitance of a surface is measured and with this exitance the surface temperatures are estimated. The cameras used in building physics measure in the range 2-13μm. The sources of error can be divided into two groups:

a- the inaccuracy of the exitance measurement
b- the inaccuracy of deriving a surface temperature from exitance.

Ad a.
The sensor used in the camera also emits longwave radiation (as all bodies do) and is in a way blinded by its own radiation. That is why in the past the sensors were (cryogenically) cooled till e.g. 80K. To date cameras without cooling are on the market with a good resolution.

Ad b.
Errors will be made if:
- the surface is a selective emitter and not Lambertian.
- the emittance and its temperature dependency are not known very well
- the reflections are not known

For a grey diffuse emitter without reflections the temperature is derived from the measured exitance:

\[ T = \left(\frac{E}{\varepsilon \sigma}\right)^{0.25} \]

So e.g. temperature measurement of a polished metal surface is not very accurate. Reflections are also troublesome. If the longwave radiation is reflected isotropic diffuse it may cause a small error for high emittance surfaces. One can often observe, however, specular reflections on pictures taken with an infrared camera, e.g. in front of glazing.

With thermography very small temperature differences can be detected but the absolute temperature is often not accurate.

### 6.3.2 Shortwave radiation

For thermal modelling the wavelength is not taken into account and absorptance, reflectance and transmittance are the mean values weighed with the solar spectrum. This leads to some inaccuracies when the radiation deviates very much from daylight (e.g. after reflection at a coloured wall).

The models for shortwave reflection and transmission can be very complicated. The most used simple models for reflection are:

- The reflected radiance is equal for all directions and independent of the angular distribution of the incident radiance. The reflection is diffuse (Lambertian).

- The reflection is specular: Specular reflection for an isotropic surface (no dependency on \( \phi \)) can be described with:

\[
L_r(\Theta, \phi) = \rho_s(\Theta) L_i(\pi - \Theta, \phi)
\]

For glazing \( \rho(\Theta) \) can be calculated with the well-known Fresnel formulas, with measured values or with empirical formulas.

For transmission through transparent glazing only direct transmission is used (transmitted radiance has the same direction as the incident one). Direct transmission is described with:

\[
L_t(\Theta, \phi) = \tau_s(\Theta) L_i(\Theta, \phi)
\]

Also here the transmission can be calculated with the Fresnel equations. An example of an empirical transmittance function for glazing is the formula of Riviero:

\[
\tau = \tau(0) (\cos \Theta + \sin^3 \Theta \cos \Theta)
\]
The transmittance and reflectance of a single glass pane can be calculated with the Fresnel equations. If the radiation is not polarized it is assumed that one half is polarized parallel (∥) to the surface and the other half perpendicular (⊥). The equations for estimating the reflectance and transmittance of a single pane for each polarization component are:

\[
\tau(\Theta_i) = \frac{x(1-y)^2}{1-y^2x^2}, \quad \rho(\Theta_i) = y\left[1 + \frac{x^2(1-y)^2}{1-y^2x^2}\right]
\]

where: \(x = \exp\left(-\frac{a d}{\cos\Theta_r}\right)\) with (Snellius): \(\sin\Theta_r = \frac{1}{n}\sin\Theta_i\)

\[
y = \begin{cases} 
\frac{(n-1)^2}{n+1} & \text{if } \Theta_i = 0 \\
\frac{\tan^2(\Theta_i - \Theta_r)}{\tan^2(\Theta_i + \Theta_r)} & \text{if } \Theta_i \neq 0 \text{ for the } \parallel \text{ component} \\
\frac{\sin^2(\Theta_i - \Theta_r)}{\sin^2(\Theta_i + \Theta_r)} & \text{if } \Theta_i \neq 0 \text{ for the } \perp \text{ component}
\end{cases}
\]

and: \(\Theta_i = \) angle of incidence  
\(\Theta_r = \) angle of refraction  
\(a d = \) optical depth, \(1/\cos\Theta_r = \) optical mass  
\(a = \) absorption coefficient of glass=0.011 10-3 [m-1]  
\(d = \) thickness of glass e.g. 4mm  
\(n = \) refraction index, e.g. glass \(n=1.52\)

After calculating for both polarization components the reflectance, transmittance and absorptance the resulting reflectance, transmittance and absorptance is just the arithmetic mean.
The solar direct transmittance $\tau_e$ of glazing can be calculated with

$$\tau_e = \frac{\int_0^\infty S_\lambda \tau(\lambda) \, d\lambda}{\int_0^\infty S_\lambda \, d\lambda}$$

and similarly the solar direct reflectance

$$\rho_e = \frac{\int_0^\infty S_\lambda \rho(\lambda) \, d\lambda}{\int_0^\infty S_\lambda \, d\lambda}$$

Where $S_\lambda$ is the solar spectrum and $\tau(\lambda)$ the spectral transmittance. This is not the same as the light transmittance as

$$\tau_{\text{visual}} = \frac{\int_{\lambda_{780\text{nm}}}^{\lambda_{380\text{nm}}} S_\lambda \tau(\lambda) V(\lambda) \, d\lambda}{\int_{\lambda_{380\text{nm}}}^{\lambda_{780\text{nm}}} S_\lambda V(\lambda) \, d\lambda}$$

where: $V(\lambda) = \text{spectral luminous efficiency}$

The luminous efficacy of solar radiation follows from:

$$K = \frac{\int_0^\infty E_{\text{sun,}V(\lambda)} \, d\lambda}{\int_0^\infty E_{\text{sun}} \, d\lambda} \approx 110 \text{ lumen / Watt}$$

This factor depends on the solar altitude, the cloudiness and the turbidity of the atmosphere. Also for artificial lighting this factor can be calculated giving the ratio of the luminous flux and the power needed for it. Incandescent light bulbs have a small efficacy value, e.g. for a blackbody with 4000K, $K=47.5$. Nowadays energy efficient lamps have a $K>>100$. 

![Spectral transmittance of glass](image)
Thermal conductivity glass $\lambda = 0.8$

<table>
<thead>
<tr>
<th></th>
<th>$\tau$</th>
<th>$\rho$</th>
<th>$\alpha$</th>
<th>$\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Float glass</td>
<td>0.826</td>
<td>0.075</td>
<td>0.099</td>
<td>0.837</td>
</tr>
<tr>
<td>Low E-glass</td>
<td>0.583</td>
<td>0.302</td>
<td>0.115</td>
<td>0.09</td>
</tr>
</tbody>
</table>

Properties of glass

Spectrum of solar radiation

Assume that the earth seen from space can be considered as a Planckian body and the reflectance of solar radiation is 0.43 (the albedo of the planet). With the energy balance (thermal equilibrium, no heating up of cooling down of the planet) the temperature at the top of the atmosphere can be calculated. The radius of the earth is $R_{\text{earth}} = 6.4 \cdot 10^3 \, \text{km} \left( \frac{40 \cdot 10^3}{2 \pi} \right)$.

$$4\pi (R_{\text{earth}})^2 \sigma T^4 = \pi(1-0.43) (R_{\text{earth}})^2 E_o \rightarrow T=244 \, \text{K} \approx -29 \, ^\circ\text{C}$$

This is about the temperature in the troposphere at a height of 6.8 km. A change of the albedo of the planet (e.g. by more clouds) would have a dramatic effect on the temperature (If the reflectance is 1% higher the temperature is 0.6°C lower). Moreover the temperature is very dependant on the sun-planet distance. So other planets will have very different temperatures.

The solar energy (power) received by the earth at the outside of the atmosphere is: $\pi(R_{\text{earth}})^2 E_o=17 \cdot 10^{18} \, \text{W}$!
The relation between the incident and the reflected radiance is given by the bi-directional reflection function:

\[
L_r(\Theta, \varphi, \lambda) = \int_{2\pi} L_i(\Theta_i, \varphi_i) f_r(\Theta, \varphi, \Theta_i, \varphi_i) \cos \Theta_i \, d\Omega_i / \pi
\]

Where: \( L_r \) = reflected radiance
\( L_i \) = incident radiance
\( f_r \) = bi-directional reflection function
other symbols: see figure

In the same way the transmission of radiation can be described with a bi-directional transmission function.

\[
L_t(\Theta, \varphi, \lambda) = \int_{2\pi} L_i(\Theta_i, \varphi_i) f_t(\Theta, \varphi, \Theta_i, \varphi_i) \cos \Theta_i \, d\Omega_i / \pi
\]

where \( L_t \) = transmitted radiance
\( f_t \) = bi-directional transmission function
other symbols: see figure

If the reflection and the transmission are ‘selective’ then the bi-directional functions depend also on the wavelength. In that case the radiance must be exchanged by the spectral radiance.

The radiance incident on a surface must equal the sum of the absorbed, reflected and transmitted one. So:

\[
\alpha(\Theta, \varphi, \lambda) + \rho(\Theta, \varphi, \lambda) + \tau(\Theta, \varphi, \lambda) = 1
\]

where:
\[
\rho(\Theta, \varphi, \lambda) = \iint_{2\pi} f_r(\Theta, \varphi, \lambda) \cos \Theta_i \, d\Omega_i / \pi
\]
\[
\tau(\Theta, \varphi, \lambda) = \iint_{2\pi} f_t(\Theta, \varphi, \lambda) \cos \Theta_i \, d\Omega_i / \pi
\]

The bi-directional reflection function for diffuse reflection is:

\[
f_r = \rho(\Theta, \varphi, \lambda)
\]
6.4 Radiation exchange in enclosures

In an enclosure the radiation leaving a surface element will be distributed over the surfaces of the enclosure by direct exchange and by (multiple) reflections.

In buildings the distances between surfaces are mostly small enough to neglect the absorption and scattering of radiation by air ($\tau = 1$).

6.4.1 Direct radiation exchange

If $\Phi_i$ is the radiant flux that leaves surface element $i$ and $\Phi_{i\rightarrow j}$ is the part of $\Phi_i$ that is directly intercepted by the surface element $j$, then this part is:

$$\Phi_{i\rightarrow j} = F_{ij} \Phi_i$$  \hspace{1cm} (6.8)

If the radiance of the radiation leaving the surface is independent of the direction (diffuse) and is the same for each place on a subsurface then the factor $F_{ij}$ depends only on the geometric orientations of surface pairs with respect to each other, i.e. the way surface $1$ ‘sees’ surface $2$. $F_{ij}$ is named the view factor or configuration factor or shape factor.

All radiation leaving $i$ is received by the surrounding surfaces.

This means: \[
\sum_{j=1}^{N} \Phi_{i\rightarrow j} = \Phi_i \quad \text{or} \quad \sum_{j=1}^{N} F_{ij} = 1
\]  \hspace{1cm} (6.9)

where $N$ is the number of the surfaces of the enclosure.

If surface $i$ has the same exitance as the surface $j$ the radiant flux from $i\rightarrow j$ must be the same as from $j\rightarrow i$ (second law of thermodynamics).

$$\Phi_{i\rightarrow j} - \Phi_{j\rightarrow i} = A_i F_{ij} M - A_j F_{ji} M = 0$$

So:

$$A_i F_{ij} = A_j F_{ji}$$  \hspace{1cm} (6.10)

This is called the view factor reciprocity relationship or reciprocity property of view factors.

For a number of geometries analytical formulas of view factors are available e.g. for parallel rectangular surfaces, perpendicular rectangular surfaces. In many thermal simulation models these factors are calculated.
In section 6.1 we derived that the radiant flux $\Phi$ leaving a surface element with area $dA_i$ and received by a face with area $dA_j$ is:

$$d^2\Phi_{i\rightarrow j} = L_i \frac{dA_i \cos \Theta_{ij} dA_j \cos \Theta_{ji}}{r_{ij}^2}$$

where:
- $\Theta_{ij} = \text{angle between normal on surface } i \text{ and the direction given by the line between surface } i \text{ and } j.$
- $L_i = \text{radiance of the radiation from surface } i.$
- $\Theta_{ji} = \text{angle between normal on surface } j \text{ and the direction given by the line between surface } i \text{ and } j.$
- $r_{ij} = \text{the distance between a point on surface } i \text{ and one on surface } j.$

After integration over the surfaces $i$ and $j$ the result is:

$$\Phi_{i\rightarrow j} = \int dA_i \int dA_j L_i dA_j \frac{\cos \Theta_{ij} \cos \Theta_{ji}}{r_{ij}^2}$$

If the radiance is independent of the direction (diffuse) and equal for each point on surface $i$ or:

$$L_i = \frac{\Phi_i}{A_i \pi}$$

then $\Phi_{i\rightarrow j} = F_{ij} \Phi_i$

with the view factor:

$$F_{ij} = \frac{1}{\pi A_i A_j} \int dA_i \int dA_j \frac{\cos \Theta_{ij} \cos \Theta_{ji}}{r_{ij}^2}$$
6.4.2 Radiation exchange with reflections

In an enclosure the radiation leaving a surface will be partially reflected by other surfaces. So the total radiation received by a surface consists of a part that is directly received from the other surfaces and a contribution from (multiple) reflections. If the reflection is diffuse and the reflected radiance the same for each place on a subsurface then similar to the view factor an exchange factor can be defined. The fraction of the total flux that leaves surface \( i \) and is received by surface \( j \) both directly and by reflection is named the exchange factor \( \psi_{ij} \):

\[
\Phi_{ij} = \psi_{ij} \Phi_i
\]

Where \( \Phi_i \) = radiation leaving surface \( i \) if there was no reflection.

The direct exchange was: \( i \rightarrow j \) \( F_{ij} \Phi_i \)

and by reflections using the definition of \( \psi_{ij} \) :

\[
i \rightarrow j \sum_{k=1}^{N} \rho_k F_k \Phi_i \psi_{kj}
\]

So the equation for \( \psi_{ij} \) is:

\[
\psi_{ij} = F_{ij} + \sum_{k=1}^{N} \rho_k F_k \psi_{kj}
\] (6.11)

One can easily proof that also for \( \psi_{ij} \) the reciprocity relation yields:

\[
A_i \psi_{ij} = A_j \psi_{ji}
\] (6.12)

The summation rule is different.
If surface \( j \) receives \( \psi_{ij}\Phi_i \) then a part will disappear by absorption and transmission. This is \( (1 - \rho_j) \psi_{ij}\Phi_i \). As all radiation from surface \( i \) will finally disappear the summation rule is:

\[
1 = \sum_{j=1}^{N} (1 - \rho_j) \psi_{ij}
\]

(6.13)

Note that by solving accurately this equation by computer the distribution of radiation is still an approximation of the physical reality. Besides the assumption of diffuse reflection, the assumption of constant radiance for each part of a subsurface is not correct. Dividing a subsurface into more subsurfaces will give a different distribution as each part receives a different radiance.

Often the exchange factor is defined differently: \( \psi_{ij}^* = (1 - \rho_j)\psi_{ij} \). With this definition the reflectance appears in the reciprocity relation but disappears in the summation rule.

There are some cases for which the equations can be solved easily. The solutions will be used in simplified equations as shall be treated in section.

### A Two radiosities model

Suppose an enclosure has but 2 different radiosities, one of surface 1 and one on 2. If the view factor \( F_{12} \) is known then all factors are known:

\[
F_{11} = 1 - F_{12}
\]

With the reciprocity relation one finds: \( F_{21} = A_1 F_{12}/A_2 \)

With the summation rule: \( F_{22} = 1 - A_1 F_{12}/A_2 \)

Equation 6.11 for two surfaces is:

\[
\begin{align*}
\psi_{11} &= F_{11} + \rho_1 F_{11} \psi_{11} + \rho_2 F_{12} \psi_{21} \\
\psi_{21} &= F_{21} + \rho_1 F_{21} \psi_{11} + \rho_2 F_{22} \psi_{21}
\end{align*}
\]

After substitution of the view factors one finds:

\[
\begin{align*}
\psi_{12} &= \frac{1}{(1 - \rho_1)(1 - \rho_2)(\frac{1}{F_{12}} + \frac{\rho_1}{1 - \rho_1} + \frac{A_1 \rho_2}{A_2(1 - \rho_2)} )} \\
\psi_{11} &= \frac{1}{(1 - \rho_1)(\frac{1}{F_{12}} + \frac{\rho_1}{1 - \rho_1} + \frac{A_1 \rho_2}{A_2(1 - \rho_2)} )}
\end{align*}
\]

(6.14)

If \( F_{12} = 1 \) and \( A_1 << A_2 \) \( \psi_{11} = 0 \) i.e. no radiation leaving surface 1 is coming back by reflection at surface 2, surface 2 is apparently ‘black’.

If \( A_1 = A_2 \) the enclosure is a cavity with two parallel surfaces. \( F_{12} = 1 \) and:

\[
\begin{align*}
\psi_{12} &= \frac{1}{1 - \rho_1 \rho_2}, & \psi_{11} &= \frac{\rho_2}{1 - \rho_1 \rho_2}
\end{align*}
\]

(6.15)
B The perfectly diffusing sphere

If the enclosure has the shaped of a sphere the radiation of a part of it is equally distributed over all other surfaces and the view factors are:

\[ F_{ij} = \frac{A_j}{\sum_k A_k} \quad (6.16) \]

N.B.: \( F_{ii} \neq 0 \): each surface 'sees' itself (is curved)

The solution of equation 6.11 is now:

\[ \psi_j = \frac{A_j}{\sum_k A_k (1 - \rho_k)} = \frac{A_j}{A_i (1 - \rho)} \quad (6.17) \]

In a hollow sphere with a high reflective coating (\( \rho \)) on the inside surface a light source is placed. In the sphere is a small hole (area \( A_h \), reflectance \( \rho = 0 \)). Calculate the irradiance on the hole.

Divide the sphere in small areas \( A_i \). Each area receives radiation \( A_i E_i \). So on the hole: \( E_{\text{tot}} = E_h + \Sigma \rho A_i E_i \psi_{ih}/A_h \). If \((1 - \rho) A_i >> \rho A_i \) then

\[ E_{\text{tot}} \approx \rho \Sigma A_i E_i / (A_i (1 - \rho)) \]

So by measuring \( E_{\text{tot}} \) the total flux \( \Sigma A_i E_i \) from the source can be estimated. The sphere is integrating the flux. That is why the perfectly diffusing sphere is also called the integrating sphere.

There are different methods to solve the equations for \( \psi_i \):

a. Iterative

If \( F_{ij} \propto \sum_{k=1}^N \rho_k F_{ik} \psi_{kj} \)

Then the first iteration is: \( \psi_{ij} \approx \psi_{ij}^{(0)} = F_{ij} \)

The second:

\[ \psi_{ij} \approx \psi_{ij}^{(0)} + \psi_{ij}^{(1)} = F_{ij} + \sum_{k=1}^N \rho_k F_{ik} F_{kj} \]

So \( \psi_{ij} = F_{ij} + \sum_{n=1}^\infty \psi_{ij}^{(n)} \) with \( \psi_{ij}^{(n)} = \sum_{k=1}^N \rho_k F_{ik} \psi_{ij}^{(n-1)} \)

The smaller \( \rho_k \) is the quicker will be the convergence. This iteration can be understood in a physical way: \( \psi_{ij}^{(n)} \) is the contribution of the \( n \)th reflection.
b. **Matrix inversion**

This takes more time on the computer. Very much time is gained by introducing variables in such a way that but one matrix inversion is needed and that the matrix is symmetric and positive definite.

If \( \psi_{ij} = A_i \psi_{ij} + \delta_{ij} A_i / \rho_i \)

\[
\delta_{ij} = \begin{cases} 
1 & i = j \\
0 & i \neq j 
\end{cases} \quad \text{(kronecker symbol)}
\]

then the equation is:

\[
\sum_{p=1}^{N} U_{ip} \psi^*_{pj} = \delta_{ij}
\]

with

\[
U_{ip} = \begin{cases} 
\rho_i / A_i & i = p \\
-\rho_i \rho_p F_{ip} / A_p & i \neq p 
\end{cases}
\]

The matrix \([U]\) is symmetric and positive definite i.e. for any \(x_i\):

\[
\sum_i \sum_j U_{ij} x_i x_j > 0
\]

After inversion \(\psi_i\) can easily be calculated

\[
\psi_{ij} = \frac{1}{A_i} \left[ U^{-1} \right]_{ij} \delta_{ij} / \rho_i
\]

### 6.5 Longwave heat transfer

#### 6.5.1 The thermal resistance for radiative heat transfer

For longwave radiation the transmittance of most materials is zero, so:

\[ \rho_k = 1 - \varepsilon_k. \]

As the emittance is large for most materials and finishing’s the reflected part is small. Because of this the presented theory for exchange is most used.

The net heat transport between two surfaces with temperatures \(T_i\) and \(T_j\) is now:

\[
\Phi_{ij} = \varepsilon_i \Phi_{j\rightarrow i} - \varepsilon_i \Phi_{i\rightarrow j} = A_i \varepsilon_i \varepsilon_j \psi_j \sigma (T_i^4 - T_j^4)
\]

This equation can be written as:

\[
\Phi_{ij} = A_i \varepsilon_i \varepsilon_j \psi_j \sigma (T_i^4 + T_j^4) (T_i - T_j) = A_i \varepsilon_i \varepsilon_j \psi_j h_{sb} (T_i - T_j) \quad (6.18)
\]

**Surface heat transfer**
Because $T_i$ and $T_j$ are thermodynamic (absolute) temperatures and because in building physics the difference is relatively small, linearization causes but an acceptable error (constant $h_{rb}$): 

$$h_{rb} \approx 5.5 + 0.055(\Theta - 16)$$

With $\varepsilon = 0.9$

$$\varepsilon h_{rb} = 5 + 0.05(\Theta - 16)$$

<table>
<thead>
<tr>
<th>$T_i(°C)$</th>
<th>$h_{rb}(\text{W/m}^2\text{K})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.61</td>
</tr>
<tr>
<td>5</td>
<td>4.87</td>
</tr>
<tr>
<td>10</td>
<td>5.14</td>
</tr>
<tr>
<td>15</td>
<td>5.41</td>
</tr>
<tr>
<td>20</td>
<td>5.70</td>
</tr>
</tbody>
</table>

For the radiant heat exchange between surface $i$ and surface $j$ a thermal resistance can be defined:

$$R_{ij} = \frac{T_i - T_j}{A_i \cdot \Phi_{ij}} = \frac{1}{A_i \cdot \varepsilon_i \varepsilon_j h_{rb} \psi_{ij} A_j} = \frac{R_{ji}}{\psi_{ij}}$$  \hspace{1cm} (6.19)

On can proof that the equations for radiative heat exchange can be exactly represented by a thermal resistances network thermal network. Below is drawn this network for three surfaces. The extension to more surfaces is straightforward but difficult to draw. The resistance $R_{0ij}$ is:

$$R_{0ij} = \frac{1}{h_{rb} \Gamma_{ij}}$$

$$\begin{align*}
\text{One can see that if for all surfaces } \varepsilon = 1 \text{ the exchange factor equals the view factor and that the influence of the emittance is represented by a resistance between the surface and the (‘delta’) network for the blackbody radiation exchange.}
\end{align*}$$
6.5.2 Radiant heat exchange in a cavity

Equation 6.15 gives the exchange for planparallel surfaces. So the heat exchange is:

$$\Phi_{12} = \frac{\varepsilon_1 \varepsilon_2}{1-(1-\varepsilon_1)(1-\varepsilon_2)} A(T_1 - T_2) h_{rb} = A(T_1 - T_2) h_t$$

The total cavity ‘surface’ coefficient for radiation yields:

$$h_t = \frac{h_{rb}}{1 + \frac{1}{\varepsilon_1} - 1}$$  \hspace{1cm} (6.20)

E.g. with $h_{rb} = 5.6 \text{ W/m}^2\text{K}$, $\varepsilon_1 = \varepsilon_2 = 0.9$: $h_t = 4.5 \text{ W/m}^2\text{K}$

And with $h_{rb} = 5.6 \text{ W/m}^2\text{K}$, $\varepsilon_2 = 0.1$ en $\varepsilon_1 = 0.9$

$$h_t = \frac{5.6}{\frac{2}{0.9} + 1} = 0.5 \text{ W/m}^2\text{K}$$

A low emittance coating on the cavity surface has a very large impact on the heat transfer coefficient.

Coatings with low emittance are applied in the cavity of double glazing, on the absorber of a solar collector etc. In a ventilated cavity in the longer term the surfaces might become dirty and the low emittance coating is not effective anymore.

Because a cavity has a depth the radiation from a point is spread somewhat over a surface area on the opposite side. This means that the heat flow density is somewhat averaged over the surface. It can be proved that 90% of the radiation is spread over a circle area with a diameter of approx. 6 x the cavity width (and 50% with 2 x the width). For narrow cavities the error is negligible. If the temperature is a linear function of the height the assumption of one resistance will be entirely correct, because the area averaged temperature equals the temperature in the centre of the area.
6.5.3 Surface coefficients for radiative heat exchange

Near a surface heat is exchanged with the air (convective) and by radiation with other surfaces. Combining the two surface coefficients is not obvious. Only when one can assume that the radiant temperature ‘seen’ by the considered surface is close to the air temperature it can be done. To estimate this coefficient the two temperature model eq.6.15 can be used. Substitution of \( \rho = 1 - \epsilon \) one finds (Christiansen. Wied. Ann.19, 1883, s.267):

\[
h_r = \frac{h_{ib}}{1/F_{12} + (1-\epsilon_1)/\epsilon_1 + (1-\epsilon_2)A_1/\epsilon_2A_2}
\]  

(6.21)

For a flat surface \( F_{12}=1 \). So with \( A_1<<A_2 \) one finds:

\[
h_r = \epsilon_1 h_{ib}
\]  

(6.22)

This result is not surprising. If the surface is not flat as in a 2-D or 3-D corner the result is different however because of \( F_{12} < 1 \); the surfaces of the corner ‘see’ each other. In order to estimate for a 2-D corner \( F_{12} \) a flat plane \( A_3 \) is imagined that contains the part of the corner that has a temperature different from the environment (see figure). The view factor \( A_3 \rightarrow A_1 \) is 1, so \( F_{13} = A_3F_{31}/A_1 = A_3/A_1 = \sin(\beta/2) \). As all radiation passing \( A_3 \) is received by the surroundings \( F_{12}=F_{13} \). The surface heat exchange from the corner to the surroundings is with eq. 6.15:

\[
h_r = \frac{\epsilon_1 h_{ib}}{\epsilon_1 / \sin(\beta/2) + 1 - \epsilon_1}
\]  

(6.23)

For \( \beta=90^\circ \) and \( \epsilon=0.9 \) the surface coefficient of the 2D corner is about 0.7 the coefficient of a flat surface. This means that if the corner is a part of an exterior wall and the temperature outdoors is low the surface temperature in the corner will be lower than the temperature of the rest of the wall. This effect is strengthened by a lower surface coefficient for convection in the corner as there is less air movement in the corner.

An analogous estimate can be made for a 3D corner. The surface \( A_3 \) of a rectangular corner is now drawn as triangle (see figure). It can be shown that \( F_{13}=\sqrt{3}/3 \) and that the surface coefficient is 0.6 the value of the surface coefficient of a flat surface. Together with a low coefficient for convection the 3D corner will often have a low surface temperature and a high relative humidity. Moulds often appear in 2D and 3D corners.
In an indoor skating rink the ice surface temperature is controlled on -10 °C. The air temperature is 10 °C. Calculate the surface temperature at the interior side of the roof.

Suppose that the height of the hall is much less than the smallest horizontal dimension and that the roof surface area is 30% larger than the ice surface area. The heat loss or gain by the roof can be neglected. The emittance of ice is 0.94, of the roof 0.8. The surface coefficient for convection at the roof surface is 2.5 W/m²K. The blackbody heat transfer coefficient $h_{bb} = 4.6 \text{Wm}^2\text{K}$.

With formula 6.21 the heat transfer coefficient for radiation between roof and ice is:

$$h_r = 4.6 \cdot \frac{0.94 \cdot 0.8}{0.8 + (1/1.3) \cdot 0.94 \cdot (1-0.8)} = 3.66 \text{W/m}^2\text{K}$$

At the ceiling the heat balance is:

$$A_{ceil}q = A_{ice}h_r(T_{ice}-T_{ceil})+A_{ceil}h_c(T_{air}-T_{ceil}) = 0$$

Or $T_{ceil}=(-10 \cdot 1 \cdot 3.66+10 \cdot 1.3 \cdot 2.5)/(3.66+1.3 \cdot 2.5) = -0.6 \degree \text{C}$

Note that because $T_{ceil}$ $T_{air}$ $h_c$ is larger than usual (air is cooled at the ceiling)

If the maximum relative humidity at the ceiling is 80%, what is the maximum relative humidity of the air?

$$\text{RH}=80 \cdot p_{sat}(-0.6)/p_{sat}(10) = 28 \%.$$ 

So drying of the outdoor air will be necessary to avoid condensation. As it is much more than $p_{sat}(-10)$ the ice layer will grow.

The low surface temperature is a problem for the roof; most of the time it is warmer outside, but sometimes colder. A careful design of the roof is necessary to avoid interstitial condensation.

### 6.5.4 The integrating sphere approximation

In section 6.4.2 a formula (eq.6.17) for the exchange in an integrating sphere was derived. Replacing the reflectance by the emittance yields for the heat flow:

$$\Phi_i = \sum_j \Phi_j = \sum_j \frac{\varepsilon_i \varepsilon_j A_j}{\sum_j (\varepsilon_i A_j)} A_i (T_i - T_j) h_{bb} = A_i (T_i - \overline{T}) h_i$$

(6.24)

with: $\Phi_i = $ nett heat flow leaving surface i

$$\overline{T} = \frac{\sum \varepsilon_j A_j T_j}{\sum \varepsilon_k A_k} \text{ and } h_r = \varepsilon_i h_{bb}$$

(6.25)
The resistance network which links all surfaces with each other has now been transformed to a network where from each surface a resistance is connected to a common node (T node). With three surfaces one would call it a delta-star transformation (see figure below). This T can be understood as the mean radiant temperature Tr of the enclosure.

The hr is of eq 6.22 is mostly used as the surface coefficient for radiation with a value of 5W/m²K. This is clearly a rough estimate. E.g.

\[ T = 285 \text{ K} \quad \varepsilon = 0.873 \text{ (glass)} \quad h_r = 4.6 \text{ W/m}^2\text{K} \]

A network of three arbitrary resistances in a delta form (see figure) can have exactly the same properties as three resistances connected by a single node (star-network). This is not possible for 4 or more arbitrary resistances. If the three delta resistances are: R_{12}, R_{13} and R_{23} then the resistances in the star network are:

\[ R_1 = \frac{R_{13} R_{12}}{R_{12} + R_{13} + R_{23}} \]
\[ R_2 = \frac{R_{23} R_{12}}{R_{12} + R_{13} + R_{23}} \]
\[ R_3 = \frac{R_{13} R_{23}}{R_{12} + R_{13} + R_{23}} \]

6.5.5 The two temperatures room model
The calculation of radiation exchange between the surfaces of an enclosure is in spite of all its simplifications still very complicated, but
with the computer no problem. However to reach the same degree of accuracy for convective heat transfer is hardly possible. In that case the whole pattern of air velocities near the surface must be known. This flow is caused by free and forced convection in the room and so but for a small part caused by local temperatures near a place on a surface. It is practically impossible to calculate the correct velocity pattern in a realistic (furnished) room. Because surface heat transfer coefficients depend on these flows also the temperature distribution is not very accurate.

Generally an entirely 'mixed' situation (one air temperature) or a known vertical temperature gradient are assumed to calculate the coefficients (frequently fixed values and independently of the whole geometry).

The same situation occurs at the calculation of heat transfer in (ventilated) cavities. The heat transfer coefficient for convection depends on the air velocity and the air velocity depends on the heat transfer. Except the heat entering and leaving the cavity by the ventilation there is also a circulation of air in the cavity: air can go down on the cold side of the cavity and rise on the warm side.

In a furnished room the accuracy of the surface coefficients is still more doubtful: furniture can influence the flow strongly and radiation exchange between surface elements can be blocked.

Manual methods to estimate the heating load use but one temperature node for the indoor climate. The incoming solar energy, the casual gains and the supplied heat are all connected to this node. The walls are connected to this node with a combined surface heat transfer coefficient for radiation and convection. For transient calculations this is too inaccurate because if the temperature is kept constant by the heating system no heat storage of for example solar heat can take place. Also the difference between air and radiant temperature, important for thermal comfort, is lacking. An important improvement is a two node model: one temperature node for the calculation of transmission losses and one temperature node for the ventilation losses. The heat gains can then be split depending on the fraction that is convective and the fraction that is radiant.

An easy way to derive this model is to assume that there are but two different surface temperatures: one of the glazing (node 1) and one for the opaque surfaces (node 2). This is a good approximation for well insulated rooms. Together with the air temperature (node 3) there are three temperatures that form a delta network for the heat transfer.

In eq. 6.14 the exchange factor for this case is given. The radiant heat flow from the window to the walls is:

$$
\Phi_{12} = \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_2 + (A_1/A_2) \varepsilon_1 (1 - \varepsilon_2)} \frac{A_1 (T_1 - T_2) h_{rb}}{A_1 (T_1 - T_2) h_{rb}} = A_1 (T_1 - T_2) h_{rb},
$$

(6.26)

So the surface coefficient from window to the walls is:
\[ h_r = \frac{h_{rb}}{1 + \frac{A_1}{e_1} \left( \frac{1}{e_2} - 1 \right)} \quad (6.27) \]

N.B. An example for which eq. 6.27 yields is a pipe-in-pipe system. The cavity between the two pipes is bordered by surface \( A_1 \) of the interior pipe and surface \( A_2 \) of the exterior pipe.

The heat balance equations for the three nodes are:

\[
\begin{align*}
\Phi_{\text{win}} &= A_1 h_{cv} (T_3 - T_1) + A_1 q_{\text{win}} + A_1 h_r (T_2 - T_1); \\
\Phi_{\text{wall}} &= A_2 h_{cv} (T_3 - T_2) + A_1 q_{\text{wall}} - A_1 h_r (T_2 - T_1); \\
\Phi_{\text{air}} &= A_1 h_{cv} (T_1 - T_3) + A_2 h_{cv} (T_2 - T_3) + \Phi_{cv}
\end{align*}
\]

where \( \Phi_{cv} \) = convective heat at the air node.

\( q_{\text{wall}} \) = radiant heat on the envelope of the enclosure.

If it is assumed that each \( m^2 \) of the enclosure receives the same amount of radiation from the sun, installation, people etc. then:

\[ q_{\text{win}} = q_{\text{wall}} = \frac{\Phi_r}{(A_1 + A_2)} \]

By transformation of this network to a star a model with two temperatures for the indoor climate arises. With some algebra one finds:

\[
\begin{align*}
\Phi_{\text{win}} &= A_1 (h_{cv} + h_r^*) (T_x - T_1) \\
\Phi_{\text{wall}} &= A_2 (h_{cv} + h_r^*) (T_x - T_2) \\
\Phi_{\text{air}} &= (A_1 + A_2) (h_{cv} + h_r^*) \left( \frac{h_{cv}}{h_r^*} (T_x - T_3) + \Phi_{cv} - \frac{h_{cv}}{h_r^*} \Phi_r \right) \\
\Phi_{\text{wall}} + \Phi_{\text{win}} + \Phi_{\text{air}} &= \Phi_{cv} + \Phi_r \quad \text{(energy conservation)}
\end{align*}
\]

With eq. 6.27 for \( h_r \) the heat transfer for radiation is:

\[
\Phi_r^* = \frac{(A_1 + A_2) e_1 e_2 h_{rb}}{A_2 e_2 + A_1 (e_1 - e_2)} \approx \varepsilon_1 h_{rb} \approx 5 \text{ W/m}^2\text{K} \quad (6.28)
\]

From the picture one can see that there is no difference between wall and window; i.e. the heat losses through envelope parts are calculated with \( T_x \) and ventilation losses with \( T_a \). The link between the two nodes depends strongly on the total surface area that exchanges heat and the surface coefficient for convection. In practice this resistance can be much smaller by the presence of furniture etc. If this resistance is very small the model is similar to the one-node model. The two temperature node model is used in the simulation program HAMBASE.
6.6 Interreflections of shortwave radiation

The calculation of the interreflections of shortwave radiation is very important for lighting. It is not fundamentally different from the radiation exchange of longwave radiation. In practice of building physics it is more complicated however,

- For shortwave radiation the transmittance is very important and for longwave not at all.
- The radiation of shortwave radiant sources (sun, lamps) is usually not diffuse.
- The reflection of shortwave radiation is less diffuse than of longwave and it is strongly dependant on wavelength (colour).
- If the shortwave radiation distribution is needed for evaluating the visual performance detailed differences are important. For thermal purposes this matters less.

Computer models based on raytracing models can handle very complex geometries but accurate data for the optical properties are often not available. Often simple models for the optical properties of the surfaces
are used e.g. a linear combination of specular and diffuse reflection, direct transmission through glazing etc.

Simpler but less accurate are models using view factors. Here two very simple models will be treated.

6.6.1 Cavities with glazing

In paragraph 6.4.2. the radiation exchange between two parallel surfaces was treated. If this cavity is between two window panes (double glazing) with transmittances $\tau_1$ and $\tau_2$ the total transmittance of this combination is with eq.6.15:

$$\tau = \tau_1 \tau_2 \psi_{12} = \frac{\tau_1 \tau_2}{1 - \rho_1 \rho_2}$$  \hspace{1cm} (6.29)

This is also true for direct solar radiation and only specular reflection.

In a flat plate (solar) collector with a cover plate with transmittance $\tau$ and an absorber with absorptance $\alpha$, the amount of absorbed solar radiation will be:

$$\Phi = A \alpha \tau \psi_{12} E_2$$

Where $E_2 = \text{de irradiance of the sun.}$

$$\rho_2 = 1 - \alpha \rightarrow \alpha \tau \psi_{12} = (\alpha \tau) = \frac{\alpha \tau}{1 - \rho (1 - \alpha)}$$  \hspace{1cm} (6.30)

The factor $(\alpha \tau)$ is called the absorption-transmittance factor of a collector.

A window can be seen as the cover plate of a solar collector. The absorber consists of the surfaces in the room behind. How much solar radiation is absorbed in this passive collector?

With eq.6.14 the exchange from window to room is:

$$\psi_{12} = \frac{1}{\alpha + \frac{A_1}{A_2} (1 - \rho) (1 - \alpha)}$$

with $A_1 =$ window area $\rho =$ reflectance of the window $1 - \alpha =$ reflectance of the interior surface of the room $A_2 =$ surface area of the ‘absorber’.

So: $$(\alpha \tau) = \frac{\alpha \tau}{\alpha + \frac{A_1}{A_2} (1 - \rho) (1 - \alpha)}$$  \hspace{1cm} (6.31)
The apparent absorptance of the room (absorber) \( \alpha_{eq} \) can be found by equating this formula with the flat plate formula eq.6.30 with \( \alpha = \alpha_{eq} \):

\[
\alpha_{eq} = \frac{\alpha}{\alpha + \frac{A_1}{A_2} (1 - \alpha)}
\]

(6.32)

E.g. \( \frac{A_1}{A_2} = \frac{1}{9}, \alpha = 0.5 \) then \( \alpha_{eq} = 0.9 \)

Because the absorber is not flat (it sees itself) it behaves like a very dark surface.

\( (\alpha_{eq} \gg \alpha) \).

If the window has more than 2 parallel panes calculation is more complicated.

Suppose \( \tau_1 \) is the transmittance of pane 1 and \( \rho_1 \) the reflectance (at the other side the same values), \( L_{ia} \) the radiance of the incoming radiation at the front side and backside respectively and \( L_{iv} \) and \( L_{ra} \) the radiance leaving the pane. So:

\[
\begin{align*}
L_{ra} &= \tau_1 L_{iv} + \rho_1 L_{ia} \\
L_{rv} &= \tau_1 L_{ia} + \rho_1 L_{iv}
\end{align*}
\]

or:

\[
\begin{bmatrix} L_{iv} \\ L_{rv} \end{bmatrix} = \begin{bmatrix} \tau_1 / \tau_1 & \rho_1 / \tau_1 \\ \rho_1 / \tau_1 & \tau_1 - \rho_1^2 / \tau_1 \end{bmatrix} \begin{bmatrix} L_{ra} \\ L_{ia} \end{bmatrix} = [A_1] \begin{bmatrix} L_{ra} \\ L_{ia} \end{bmatrix}
\]

(6.33)

\( L_{ra} \) equals the incoming radiance on the next pane and \( L_{ia} \) the radiance coming from the next pane. The calculation of the transmittance of an arbitrary number of transparent panes is now not more than the multiplication of the matrices of each individual pane:

\[
\begin{bmatrix} L_{iv}^{(1)} \\ L_{rv}^{(1)} \end{bmatrix} = [A_1][A_2][A_3]...[A_n] \begin{bmatrix} L_{ia}^{(n)} \\ 0 \end{bmatrix}
\]

(6.34)

This method can also be used for the case \( \tau \) and \( \rho \) are matrices. If the Fresnel equations are used for \( \tau \) and \( \rho \) the calculation much be done for both polarization components.

A part of the solar radiation incident on a window will be absorbed by the glazing. This will increase the temperature of the panes and a part of this heat will enter the interior by the surface coefficients for radiation and convection. The fraction of the incident solar radiation at the outside entering the building by transmitted solar radiation (shortwave) and by absorbed radiation is called the g-value. The part of the heat entering that is convective is called the convective fraction. So with the
g-value the solar gain is calculated and with the convective fraction the heat that is warming directly the air (the radiant heat has to be absorbed first by indoor surfaces before it can heat the air). A glazing system with Venetian blinds at the inside will have the same direct solar transmittance as with the blinds at the outside. By the absorbed heat in the blinds the inside blinds will have a much higher g-value (the heat is already inside) and are not very effective against overheating. Also part of the air will be heated directly by the high convective fraction. By radiant heat the air is heated indirectly via the envelope. The latter is always more slow as the envelope has some heat storage capacity.

To clarify this, an example of single glazing is the most simple.

Suppose 80% is transmitted and 10% of the incident radiation is absorbed by the glazing. If $h_i = 7.5 \text{ W/m}^2\text{K}$, $h_e = 20\text{ W/m}^2\text{K}$ and the thermal resistance of the glazing can be neglected then:

$$g = 0.8 + 0.1 \times \frac{7.5}{7.5+20} = 0.83.$$

If $h_r$ at the inside is $5\text{ W/m}^2\text{K}$ then the convective fraction is

$$\frac{(0.1 \times 2.5/27.5)/0.83}{0.01}$$

If there are more window panes the first step is the calculation of the absorbed radiation in each pane e.g. with eq.6.34. The second step is to determine the components of a resistance network of the window system. Finally the heat to the interior can be calculated. For this problem software with databases with different glazing types, sun protections etc are on the market.

**6.6.2 Split flux room models**

Split flux models are used especially for estimating illuminance levels in a room on the working plane. The radiant flux entering the room either by the window (solar) or by sources in the rooms (artificial lighting) is split in a part entering in downward ($A_1 E_1$) and a part entering in upward direction ($A_2 E_2$).

If the reflection is diffuse and spread out uniformly over $A_1$ and $A_2$ the irradiance of a point $p$ on a surface of the lower part of the room is (eq6.15):

$$E_{\text{down}} = E_p + \frac{\rho_1 A_1 E_1 \psi_{11} + \rho_2 A_2 E_2 \psi_{21}}{A_1}$$

$E_p$ is the direct irradiation on point $p$ without reflections. This can be calculated separately or can be approximated by $E_1$. The second term at the r.h.s. is the internally reflected radiation.

The irradiance on a surface of the upper part of the room is accordingly:

$$E_{\text{up}} = E_2 + \frac{\rho_1 A_1 E_1 \psi_{12} + \rho_2 A_2 E_2 \psi_{22}}{A_2}$$

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If the floor area is large compared to the height the room looks like a cavity, so $A_1 = A_2$ and (eq.6.17):

$$\psi_{12} = \psi_{21} = \frac{1}{1-\rho_1\rho_2}, \quad \psi_{11} = \frac{\rho_2}{1-\rho_1\rho_2}, \quad \psi_{22} = \frac{\rho_1}{1-\rho_1\rho_2}$$

So:

$$E_{\text{down}} = E_p + \frac{\rho_1 \rho_2 E_1 + \rho_2 E_2}{1-\rho_1\rho_2} \quad (6.37)$$

$$E_{\psi} = E_2 + \frac{\rho_1 E_1 + \rho_1 \rho_2 E_2}{(1-\rho_1\rho_2)} \quad (6.38)$$

This equation is often used for artificial lighting ($E_2 = 0$)

If the room is more like a cube the integrating sphere approximation is better (eq.6.17):

$$\psi_j = \frac{A_j}{A_1(1-\overline{\alpha})}$$

So:

$$E_{\text{down}} = E_p + \frac{\rho_1 A_1 E_1 + \rho_2 A_2 E_2}{A_1(1-\overline{\alpha})} \quad (6.39)$$

$$E_{\psi} = E_2 + \frac{\rho_1 A_1 E_1 + \rho_2 A_2 E_2}{A_1(1-\overline{\alpha})} \quad (6.40)$$

Equation 6.39 is often named the BRS split flux equation. If $E_1$ and $E_2$ are the solar radiation transmitted by a window, $E_1$ could be the direct solar radiation and a part of the diffuse and $E_2$ the externally reflected radiation plus a part of the diffuse solar radiation.
Questions

In order to save energy a house-owner paints the heating radiators black. Give at least two reasons why this will not work.

In the morning there are often water droplets on cars parked outside. If they are parked close to a wall or under a tree it seldom happens. Why?

At firing pottery in a kiln t craftsmen could see from the colour of the material whether the temperature was correct. How is that possible and what was the minimum temperature of the kiln?
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Appendix A: Saturation vapour pressure

Saturation water vapour pressure $p_{sat}$ (Pa) and maximal humidity ratio $x_{sat}$ ($\Theta$) in g per kg dry air

<table>
<thead>
<tr>
<th>$\Theta$ (°C)</th>
<th>$p_{sat}$</th>
<th>$x_{sat}$</th>
<th>$\Theta$ (°C)</th>
<th>$p_{sat}$</th>
<th>$x_{sat}$</th>
<th>$\Theta$ (°C)</th>
<th>$p_{sat}$</th>
<th>$x_{sat}$</th>
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Appendix B: Quantities and symbols

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<th>Symbol</th>
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<th>Property</th>
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<tbody>
<tr>
<td>A</td>
<td>m²</td>
<td></td>
<td>Area, (oppervlakte)</td>
</tr>
<tr>
<td>V</td>
<td>m³</td>
<td></td>
<td>Volume</td>
</tr>
<tr>
<td>r</td>
<td>m</td>
<td></td>
<td>Radius (Straal)</td>
</tr>
<tr>
<td>h</td>
<td>m</td>
<td></td>
<td>Height (Hoogte)</td>
</tr>
<tr>
<td>t</td>
<td>s</td>
<td></td>
<td>Time (tijd)</td>
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<tr>
<td>D</td>
<td>m</td>
<td></td>
<td>Diameter</td>
</tr>
<tr>
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<td>m</td>
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<td>Thickness (dikte)</td>
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<tr>
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<td>Vertical coordinate</td>
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<tr>
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<td>m</td>
<td></td>
<td>Length (lengte)</td>
</tr>
<tr>
<td>b</td>
<td>m</td>
<td></td>
<td>Width, breadth, (breedte)</td>
</tr>
<tr>
<td>d*</td>
<td>m</td>
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<td>Penetration depth (dempingsdiepte)</td>
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<tr>
<td>d_eff</td>
<td>m</td>
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<td>Effective thickness</td>
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<td>t₀</td>
<td>s</td>
<td></td>
<td>Period (periodetijd)</td>
</tr>
<tr>
<td>Δt</td>
<td>s</td>
<td></td>
<td>Delay, time lag (vertraging)</td>
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<tr>
<td>D_h</td>
<td>m</td>
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<td>Hydraulic diameter (hydraulische diameter)</td>
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<td>ω/2π</td>
<td>1/s</td>
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<td>Frequency (frequentie), ω=angular frequency</td>
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<tr>
<td>symbol</td>
<td>unity</td>
<td>property</td>
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<tr>
<td>a</td>
<td>m²/s</td>
<td>Thermal diffusivity (temperatureververpenningscoëfficiënt)</td>
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<td>b</td>
<td>kg/m²Ks¹/²</td>
<td>Thermal effusivity (warmtepenetratiegetal van contactcoëfficiënt)</td>
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<td>C</td>
<td>J/Km²</td>
<td>Surface heat capacitance (warmtecapaciteit per m²)</td>
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<tr>
<td>c₀</td>
<td>J/kgK</td>
<td>Specific heat capacity of a dry material (soortelijke warmte, droog)</td>
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<tr>
<td>cₜ</td>
<td>J/kgK</td>
<td>Specific heat capacity of water (de soortelijke warmte van water)</td>
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<tr>
<td>cₚ</td>
<td>J/m³</td>
<td>Volumic internal energy (inwendige energie per volume-eenheid)</td>
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<td>qᵢ</td>
<td>W/m</td>
<td>Linear density of heat flow rate (lineaire warmtestroomdichtheid)</td>
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<td>f</td>
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<td>Temperature ratio (de binnenoppervlaktetemperatuurfactor)</td>
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<tr>
<td>Gd</td>
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<td>Degree-days (graaddagen)</td>
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<td>Gh</td>
<td></td>
<td>Degree-hours (graaduren)</td>
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<tr>
<td>h</td>
<td>J/kg</td>
<td>Specific enthalpy (soortelijke enthalpie)</td>
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<tr>
<td>H</td>
<td>J</td>
<td>Enthalpy (enthalpie)</td>
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<tr>
<td>hₑᵥ</td>
<td>W/m³K</td>
<td>Surface coefficient of convective heat transfer (warmteoverdrachtscoëfficiënt v. convectie)</td>
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<tr>
<td>hᵣ</td>
<td>W/m³K</td>
<td>Surface coefficient of radiative heat transfer (warmteoverdrachtscoëfficiënt voor straling)</td>
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<tr>
<td>hₑᵥᵣ</td>
<td>W/m³K</td>
<td>Combined surface coefficient (totale warmteoverdrachtscoëfficiënt)</td>
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<tr>
<td>Iₙclo</td>
<td>clo</td>
<td>Clothing value (1clo=0.155 m²K/W)</td>
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<tr>
<td>L</td>
<td>J</td>
<td>Latent heat (latente warmte)</td>
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<tr>
<td>Iₑ</td>
<td>J/kg</td>
<td>Specific latent heat of evaporation (soortelijke verdampings- of condensatiewarmte (hₑ))</td>
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<tr>
<td>Iₑₙ</td>
<td>J/kg</td>
<td>Specific latent heat of melting (soortelijke smelt- of stollingswarmte (hₑₙ))</td>
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<tr>
<td>P</td>
<td>W</td>
<td>Power (vermogen)</td>
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<tr>
<td>PPD</td>
<td>%</td>
<td>Predicted percentage of dissatisfied</td>
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<tr>
<td>PMV</td>
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<td>Predicted mean vote</td>
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<td>q</td>
<td>W/m²</td>
<td>Density of heat flow rate (warmtestroomdichtheid)</td>
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<tr>
<td>Q</td>
<td>J</td>
<td>Quantity of heat (hoeveelheid warmte)</td>
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<tr>
<td>R</td>
<td>m²K/W</td>
<td>Thermal resistance (warmte weerstand)</td>
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<tr>
<td>Rₑ</td>
<td>m²K/W</td>
<td>Thermal resistance of a structure (warmte weerstand van een constructie)</td>
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<tr>
<td>T</td>
<td>K</td>
<td>Thermodynamic (absolute) temperature (thermodynamische temperature)</td>
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<td>U</td>
<td>W/m²K</td>
<td>Total thermal transmittance (air to air) (warmte doororganstcosystemficiënt l.o.l.)</td>
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<tr>
<td>Z</td>
<td>m²K/W</td>
<td>Impedance (impedantie)</td>
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<td>α</td>
<td>K⁻¹</td>
<td>Linear thermal expansion coefficient (lineaire uitzettingscoëfficiënt)</td>
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<td>Φ</td>
<td>W</td>
<td>Heat flow rate (warmtestroom)</td>
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<td>λ</td>
<td>W/mK</td>
<td>Thermal conductivity (warmtegeleidingscoëfficiënt)</td>
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<tr>
<td>η</td>
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<td>Utilisation factor (benuttingsfactor) (s = solar, i = casual)</td>
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<tr>
<td>ρ</td>
<td>kg/m³</td>
<td>Density (soortelijke massa of massadichtheid)</td>
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<tr>
<td>Θ</td>
<td>°C</td>
<td>Temperature (temperature)</td>
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</tr>
<tr>
<td>Θₘᵣt</td>
<td>°C</td>
<td>Mean radiant temperature</td>
<td></td>
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</table>
A \text{ kg/m}^2\text{s}^{1/2} \quad \text{Water sorption coefficient (waterabsorptiecoëfficiënt)}

a_v \text{ m}^2/\text{s} \quad \text{(water) vapour diffusivity (} D_w \text{ in hygroscopic range) (dampverefeningscoëfficiënt)}

B \text{ m}/\text{s}^{1/2} \quad \text{Water penetration coefficient (waterpenetratiecoëfficiënt)}

b_v \text{ s}^{3/2}/\text{m} \quad \text{Water vapour effusivity (damppenetratiecoëfficiënt)}

\rho_v, C_v \text{ kg/m}^3 \quad \text{Water vapour concentration (waterdamconcentratie) (v)}

D \text{ m}^2/\text{s} \quad \text{Diffusion coefficient (Diffusiecoëfficiënt (waterdampverefeningscoëfficiënt van lucht))}

D_T \text{ kg/mKs} \quad \text{Thermal diffusion coefficient (thermische vochtgeleidingscoëfficiënt)}

D_w \text{ m}^2/\text{s} \quad \text{Moisture diffusivity (vochtverefeningscoëfficiënt)}

\text{g kg/m}^3\text{s} \quad \text{Density of moisture flow rate (vochstroomdichtheid)}

G \text{ kg/s} \quad \text{Moisture flow rate (vochtstroom)}

p_v \text{ Pa} \quad \text{Partial water vapour pressure (partiële waterdampdruk)}

p_{sat} \text{ Pa} \quad \text{Saturation water vapour pressure (verzadigingsdampspinning)}

k_m \text{ s} \quad \text{Moisture (conductivity) permeability (hydraulische(vocht)geleidings-of permeabiliteit)}

m \text{ Kg} \quad \text{(Quantity of) Mass (hoeveelheid massa)}

m_v \text{ Kg} \quad \text{Mass of vapour (massa damp)}

p_c, s \text{ Pa} \quad \text{Capillary pressure (suction) (capillaire druk (zuiging))}

R_a \text{ J/kgK} \quad \text{Gas constant of dry air (voor droge lucht (} R_a = 287.1 \text{ J/kgK) }}

R_v \text{ J/kgK} \quad \text{Gas constant of water vapour (gasconstante voor waterdamp) (} R_v = 462 \text{ J/kgK) }}

S \text{ Degree of saturation (verzadigingsgraad)}

u \text{ kg/kg} \quad \text{Moisture content mass by mass (vochtgehalte (massa per massa))}

w \text{ kg/m}^3 \quad \text{Moisture content (mass by volume) (vochtgehalte (massa per vol))}

W \text{ kg/m'sPa} \quad \text{Moisture transfer coefficient (vochtdoorgangscoëfficiënt) (1/Z)}

x \text{ kg/kg} \quad \text{Humidity (water vapour) ratio (waterdampgehalte (absolute vochtigheid))}

Z \text{ m}^2/\text{sPa/}kg \quad \text{Vapour resistance related to pressure (m/s) (dampweerstand gerelateerd aan druk) }

\beta \text{ kg/m'sPa} \quad \text{Surface coefficient of vapour transfer (s/m) (dampovergangscoëfficiënt (voor druk))}

\beta_x \text{ kg/m's} \quad \text{Surface coefficient of vapour transfer related to x (dampovergangscoëfficiënt (voor x))}

\varepsilon \text{ -} \quad \text{ =R_s/R_a=0.622}

\varepsilon_m \text{ -} \quad \text{Linear moisture expansion coefficient (relatieve lengteverandering door vocht)}

\delta \text{ s} \quad \text{Vapour permeability coefficient (dampgeleidingscoëfficiënt) } \delta_v=1.8\cdot10^{-10} \text{(s)}

\varphi \text{ -} \quad \text{Relative humidity (relatieve vochtigheid) (RH=\varphi 100\%)}

\zeta \text{ kg/m}^3 \quad \text{Specific moisture capacity related to RH (hygroscopische vochtdifferentiecapaciteit)}

\Xi \text{ kg/m}^3\text{Pa} \quad \text{Specific moisture capacity related to suction (capillaire vochtdifferentiecapaciteit)}

\Psi_{fs}, \Psi_s \text{ m}^3/\text{m}^3 \quad \text{Open porosity (open porositeit), Air content (porosity) (luchtgehalte)}

\Psi \text{ m}^3/\text{m}^3 \quad \text{Moisture content vol by vol (vochtgehalte (vol per vol)) (volume\%=\Psi 100\%)}

\sigma \text{ N/m} \quad \text{Surface tension (appervlaktespanning) (water } \sigma=75\cdot10^{-3} \text{N/m))}

\mu \text{ -} \quad \text{(Vapour) diffusion resistance factor (vochtweerstandsgetal (diffusievoorbeperking))}

\mu_d \text{ m} \quad \text{Vapour diffusion thickness (vochtweerstansdikte (diffusie))}

\rho_w \text{ kg/m}^3 \quad \text{Density of water (soortelijke massa van water, (massa) dichtheid van water)}

\theta \text{ -} \quad \text{Contact-angle (contacthoek)}

\Theta_w \text{ °C} \quad \text{Wet-bulb temperature (natte boltemperature)}

\Theta_d \text{ °C} \quad \text{Dew-point temperature (dauwpuntstemperatuur)}
<table>
<thead>
<tr>
<th>symbol</th>
<th>unity</th>
<th>property</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>Kg/m³</td>
<td>Concentration (concentratie)</td>
</tr>
<tr>
<td>C_A</td>
<td>m³/m²²Pa¹/n</td>
<td>(specifieke luchtdoorlatendheidsoëfficiënt per m²)</td>
</tr>
<tr>
<td>C_D</td>
<td>-</td>
<td>(wind)drukfactor</td>
</tr>
<tr>
<td>C_l</td>
<td>m³/ms²Pa¹/n</td>
<td>(specifieke airdoorlatendheidsoëfficiënt per m naad of kier)</td>
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<td>C_tot</td>
<td>m³/sPa¹/n</td>
<td>Flow coefficient (totale doorlatendheidsoëfficiënt)</td>
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<tr>
<td>g</td>
<td>m²/s²</td>
<td>Acceleration of free fall (versnelling van de zwaartekracht)</td>
</tr>
<tr>
<td>k_a</td>
<td>s</td>
<td>Air permeability (luchtdoorlatendheid of luchtpermeabiliteit) (k=ν k_a)</td>
</tr>
<tr>
<td>K</td>
<td>m³/m²²sPa</td>
<td>Air permeance (luchtdoordoorstroomsoëfficiënt) (n = 1)</td>
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<tr>
<td>n</td>
<td>h⁻¹</td>
<td>Ventilation rate (ventilatievoud)</td>
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<tr>
<td>n</td>
<td>-</td>
<td>Flow exponent (stromingsexponent) 1 &lt; n &lt; 2</td>
</tr>
<tr>
<td>m, q_m</td>
<td>kg/s</td>
<td>Mass flow rate (massastroom)</td>
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<td>MAC</td>
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<td>V, q_v</td>
<td>m³/s</td>
<td>Air flow rate, ventilation rate (luchtvolumestroom, volumedebiet)</td>
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<td>Pa s</td>
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<td>View factor (geometrische factor, vormfactor, blikfactor)</td>
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<td>g-value</td>
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<td>Solar heat gain factor (SHGF) (absolute zonotoetredingsfactor, ZTA)</td>
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<td>Stefan-Boltzmann constant $5.67 \cdot 10^{-8}$</td>
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<td>Å, µm, nm</td>
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Appendix C: Heat And Moisture Transfer Equations

\[ c_w = 4187 \text{ J/kgK}, \quad c_v = 1926 \text{ J/kgK}, \quad c_a = 1000 \text{ J/kgK}, \quad p_{atm} = 103325 \text{ Pa}, \]
\[ \rho_w = 1000 \text{ kg/m}^3, \quad \rho_a = 1.2 \text{ kg/m}^3, \quad R_v = 462 \text{ J/kgK}, \quad R_a/R_v = 0.622 \]
\[ L = 2.491 \times 10^5 \text{ J/kg}, \quad x = 0.62 \times 10^{-5} \rho_v \]

\[ w = w(\sigma^p c_w) \text{ with } \sigma^p = \sigma(T_0)/\sigma(T) \approx (75 - 0.17\Theta_0)/(75 - 0.17\Theta) \]
\[ \xi = (dw/dp)_T \approx \xi(\sigma^p c_w), \quad \Xi = \exp(\sigma^p c_w/R_v T_0) \xi / \rho_w R_v T_0 \]

Ideal gas law: \[ p = \rho R T \]

Kelvin: \[ p_v = p_{sat}(T) \exp(p/(\rho_R T)) \]

Clapeyron: \[ d(p_{sat}/dT) = p_{sat} L/(R_v T^2) \]

Boussinesq: \[ \rho_a(T) - \rho_a(T_0) = - \rho_a(T_0) (T - T_0)/T_0 \]

Lewis: \[ \beta = 0.62 \times 10^{-5} h c_v / c_a \]

\[ f_T = f_p = 1 \text{ or } f_T = f_p = \mu / \mu_0, \quad \mu_0 = \text{‘dry cup’} \mu \text{ or } f_T \neq f_p = ?? \]

\[ \bar{q}_w = \bar{q}_a = \bar{q} = -a_{11} \nabla T - a_{12} \nabla \rho \]

\[ \bar{g}_w = \bar{g}_a = -a_{21} \nabla T - a_{22} \nabla \rho - k_m (\bar{g}_w \bar{c}_w + \nabla p_{tot}) \]

\[ \bar{g}_a = -k_a (\nabla p_{tot} + \rho_a(T - T_0) \bar{g}_w / T_0) \]

\[ k_v = \left( \frac{\delta_v}{\mu} \right) \frac{\rho_v}{\rho_w R_v T} \]

\[ \bar{c}_z = (0,0,1) \]

\[ a_{22} = k_m + k_v \quad a_{21} = f_T k_v L \frac{\rho_w}{T} \]

\[ a_{11} = \lambda + La_{21} \quad a_{12} = f_p k_v L \]

\[ C_{\theta\theta} \frac{\partial \Theta}{\partial t} + C_{\rho \theta} \frac{\partial c_v}{\partial t} = -\nabla \cdot \bar{q}_w c_w + \nabla \cdot \bar{g}_w \nabla \Theta - \bar{g}_a \cdot (c_a \nabla \Theta + LVx) \]

\[ C_{\rho \theta} \frac{\partial \rho}{\partial t} + C_{pp} \frac{\partial p}{\partial t} = -\nabla \cdot \bar{g}_w - \bar{g}_a \cdot \nabla x \]

\[ \nabla \cdot (\bar{g}_w + \rho_a \bar{g}_w) = 0 \]

\[ C_{pp} = \Xi \sigma \]

\[ C_{\theta \rho} = 0 \]

\[ C_{\rho \rho} = \Xi p_c \left( \frac{\partial \sigma^p}{\partial T} \right) = \frac{0.17 p_c}{75 - 0.17 \Theta} C_{pp} \]

\[ C_{\theta \theta} = \rho c + wc_w \]

The boundary conditions are:
\[ q_s = h_{cv+} (\Theta_1 - \Theta_2) + h_{\rho (\Theta_2 - \Theta_3)} + g_{rain} c_w (\Theta_{rain} - \Theta_3) + L \beta_+ (p_{vi} - p_{vs}) \]
\[ g_{ws} = \beta_+ (p_{vi} - p_{vs}) + g_{rain} \]

If \( g_{as} \) is into the surface \( \beta_+ = \beta + g_{as} \cdot 0.62 \cdot 10^{-5} \), \( h_{cv+} = h_{cv} + g_{as} c_a \) else \( \beta_+ = \beta, \ h_{cv+} = h_{cv} \)

\( g_{as} \) can be derived from the solution of the equation for air transfer with boundary conditions \( p_{tot1} \) and \( p_{tot2} \).